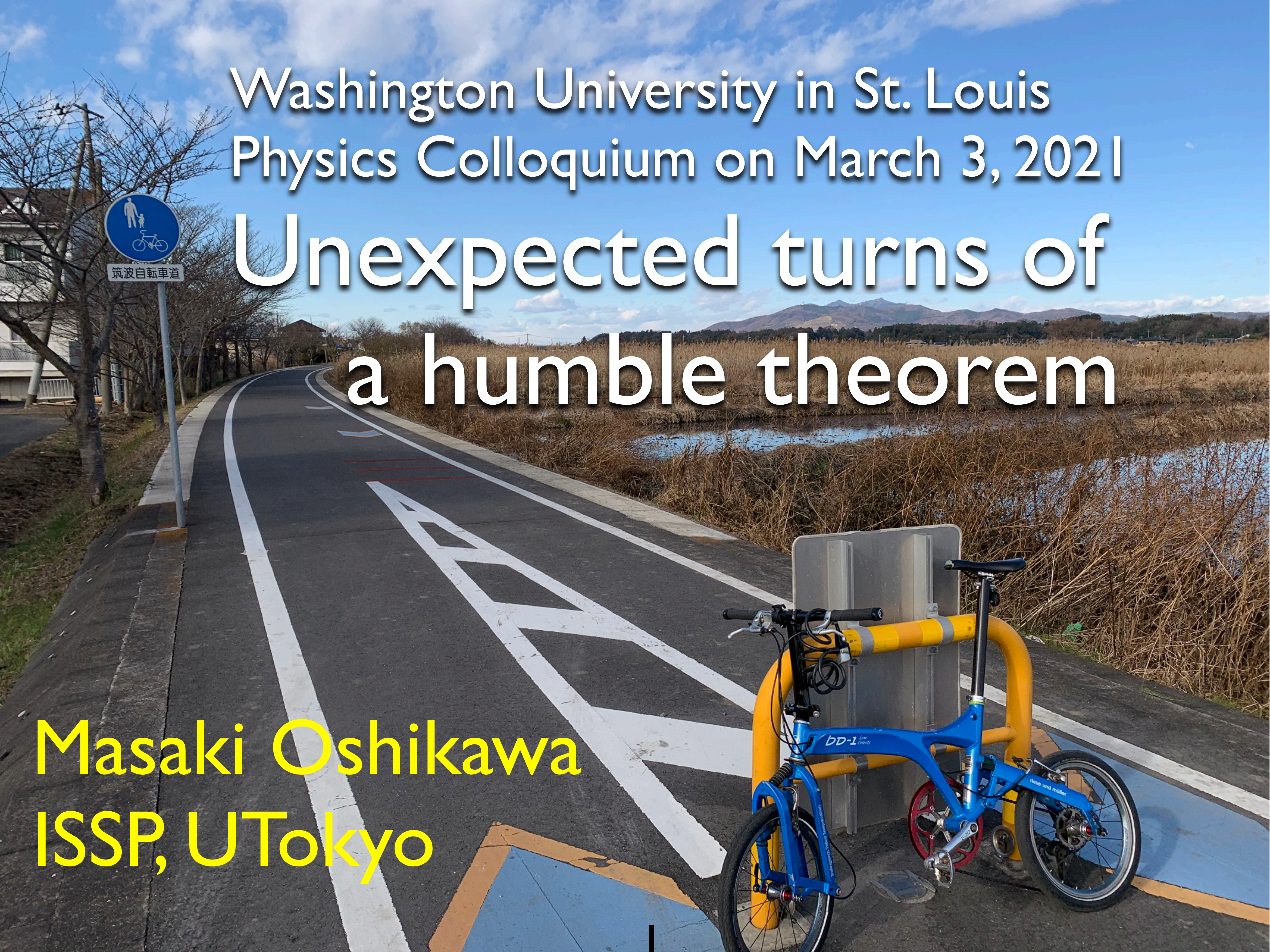
Washington University in St. Louis
Physics Colloquium on March 3, 2021

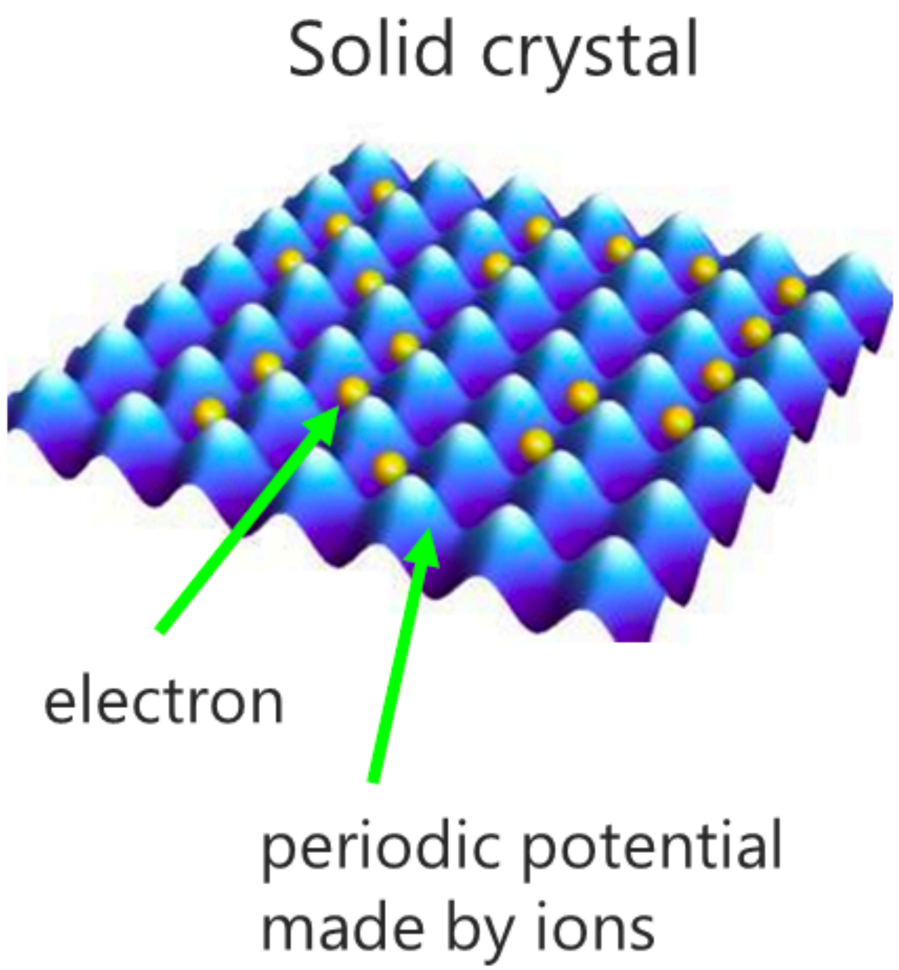
Unexpected turns of a humble theorem

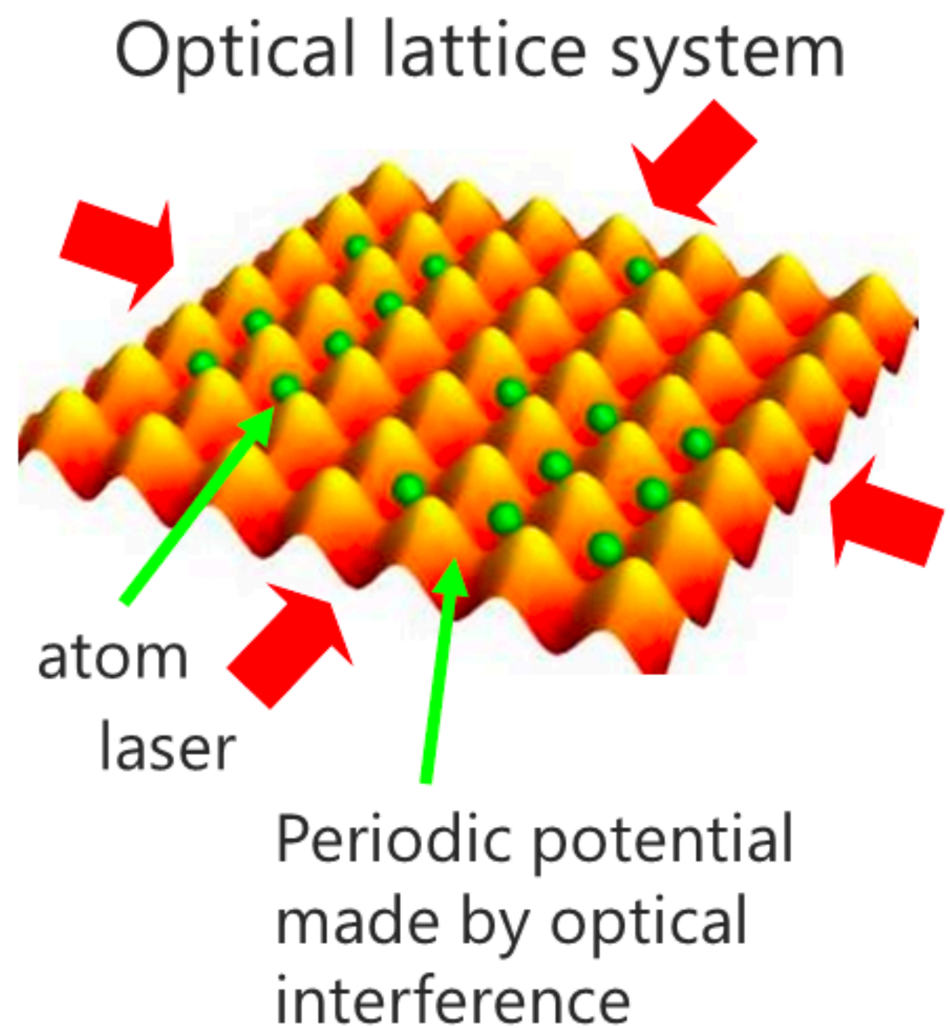
Masaki Oshikawa
ISSP, UTokyo



Quantum Many-Body Systems

From Kozuma Group (Tokyo Tech) web page





(spinless) Hubbard model

$$\mathcal{H} = \sum_{\langle j,k \rangle} \left[-t \left(c_j^\dagger c_k + c_k^\dagger c_j \right) + V n_j n_k \right]$$

It's a hard problem!

Fermion: each site is either empty (0) or occupied (1)

Λ sites: Hamiltonian is $2^\Lambda \times 2^\Lambda$ matrix

huge even for moderately large size Λ

cf. quantum computing

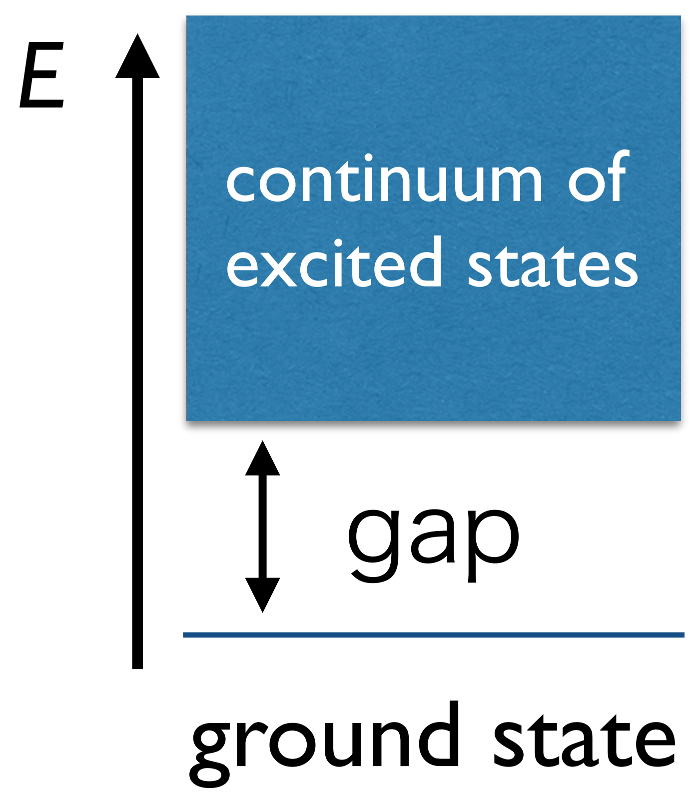
Numerical algorithms: great advancements (Quantum Monte Carlo, Density-Matrix Renormalization Group, Tensor Network, …) but still challenging

Exact solution: available only for the standard model in 1D (no longer exactly solvable in 2D and higher, or by inclusion of next-nearest-neighbor coupling etc.)

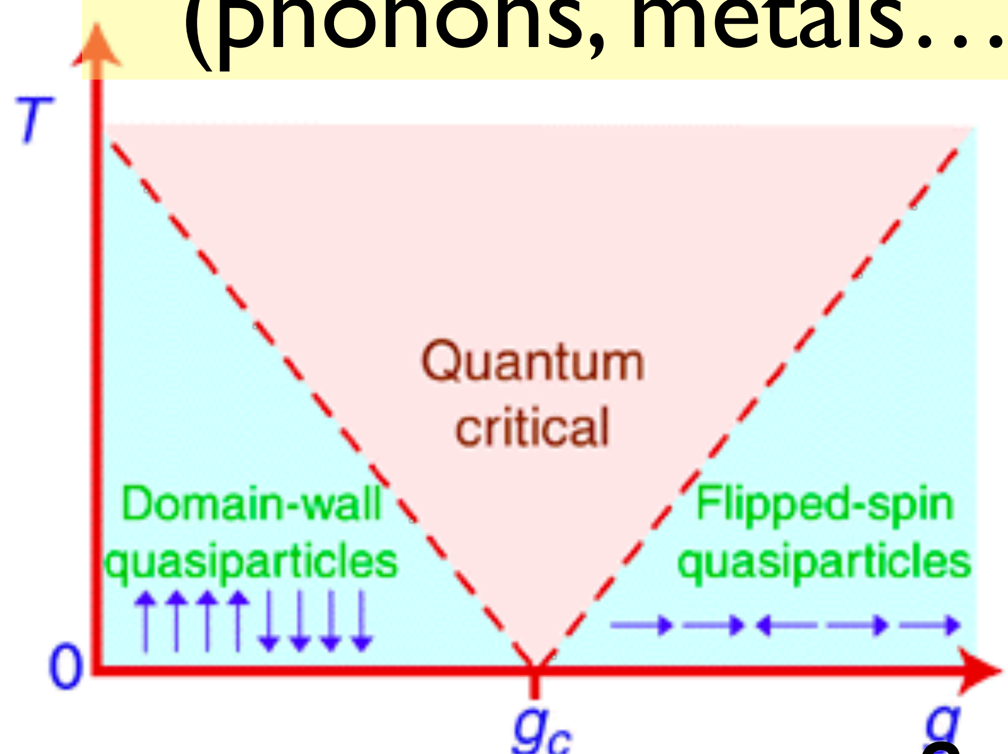
Quantum Many-Body Systems

Quantum fluctuations can drive the system at T=0 into different quantum phases, and cause quantum phase transitions between quantum phases

gapped (off-critical)



But there are gapless systems without any apparent fine-tuning… (phonons, metals…)



gapless (critical)

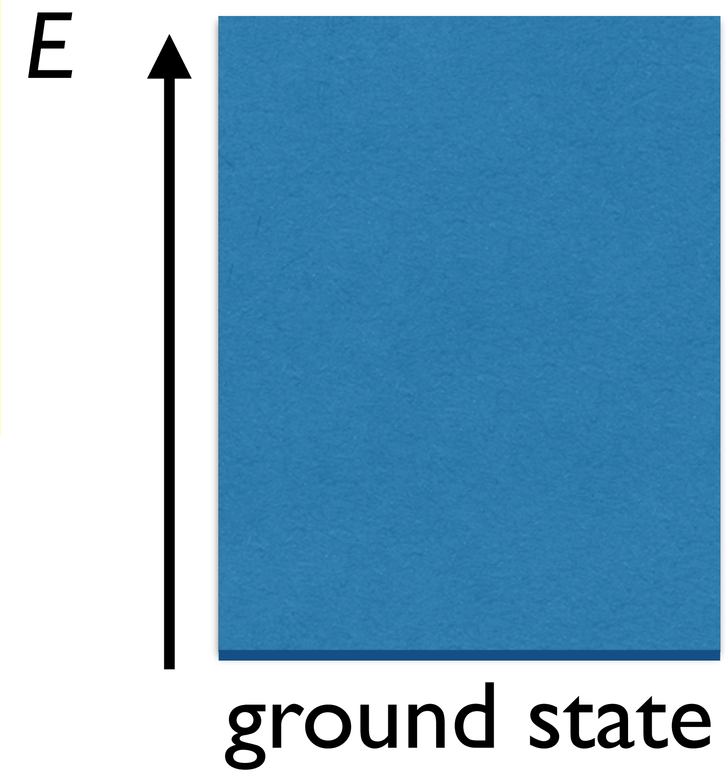


fig. by Subir Sachdev

General Principles?

Symmetries of the model

$$\mathcal{H} = \sum_{\langle j,k \rangle} \left[-t \left(c_j^\dagger c_k + c_k^\dagger c_j \right) + V n_j n_k \right]$$

U(1) symmetry

$$c_j \to e^{i\theta} c_j$$

$$c_j^\dagger \to e^{-i\theta} c_j^\dagger$$

$$n_j \equiv c_j^\dagger c_j \to n_j$$

particle number

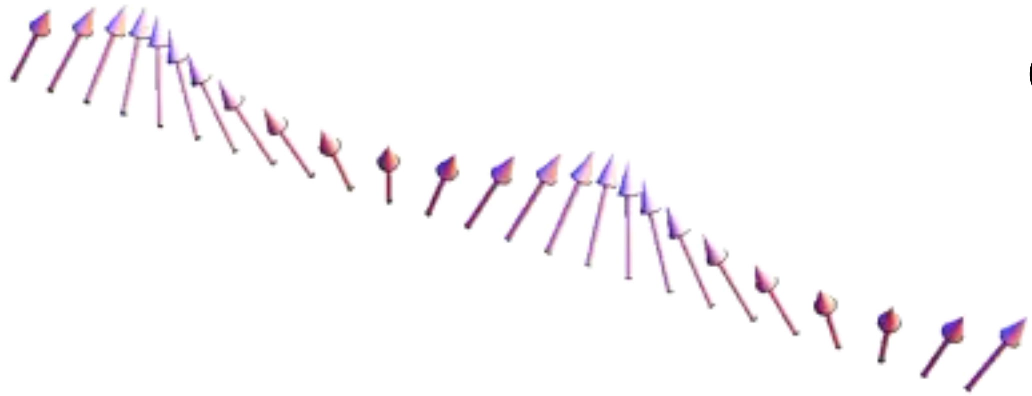
$$N \equiv \sum_j n_j$$

conserved

Noether's theorem

Can we say something about the energy spectrum?

Nambu-Goldstone Theorem



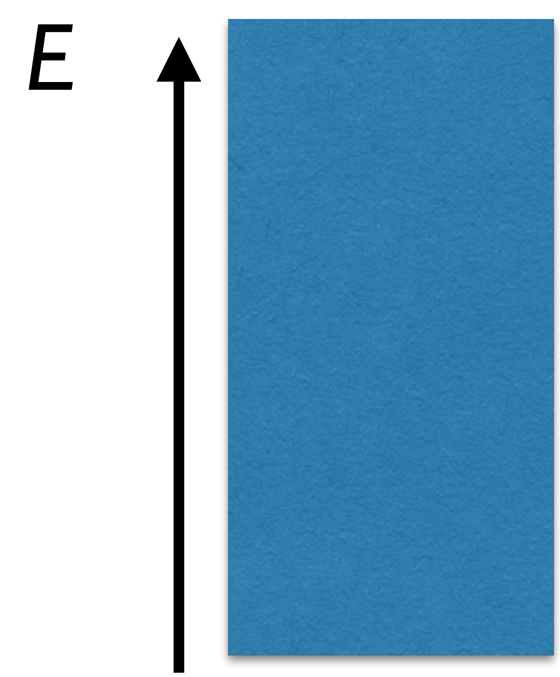
e.g. spin waves

Spontaneous breaking of a continuous symmetry (e.g. U(1))

"slow twist"

Gapless excitations

gapless (critical)

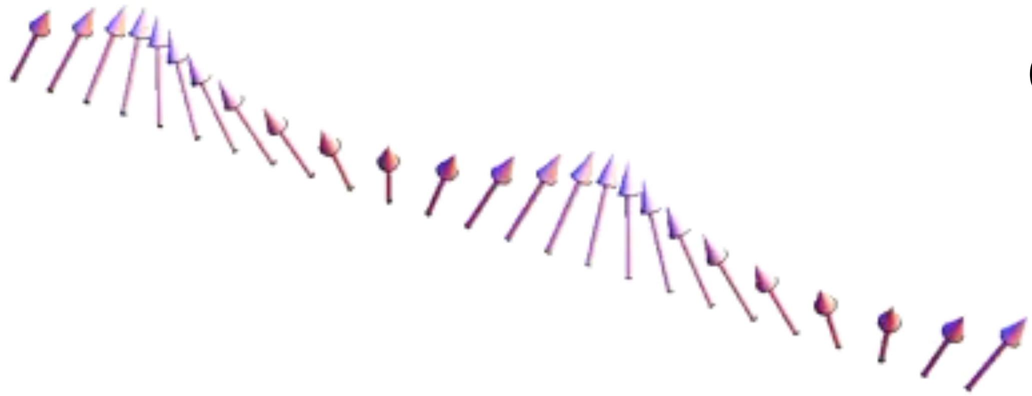


There are many gapless systems without a SSB (metals, etc.), however.

Any other mechanism for gaplessness?

yes, if there is also a lattice translation invariance

Nambu-Goldstone Theorem



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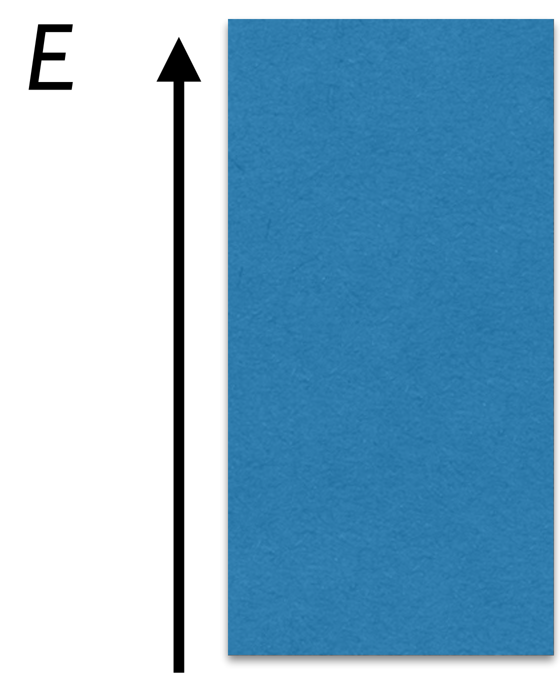

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Lieb-Schultz-Mattis Theorem in 1D

Lieb-Schultz-Mattis 1961, M.O.-Yamanaka-Affleck 1997,…

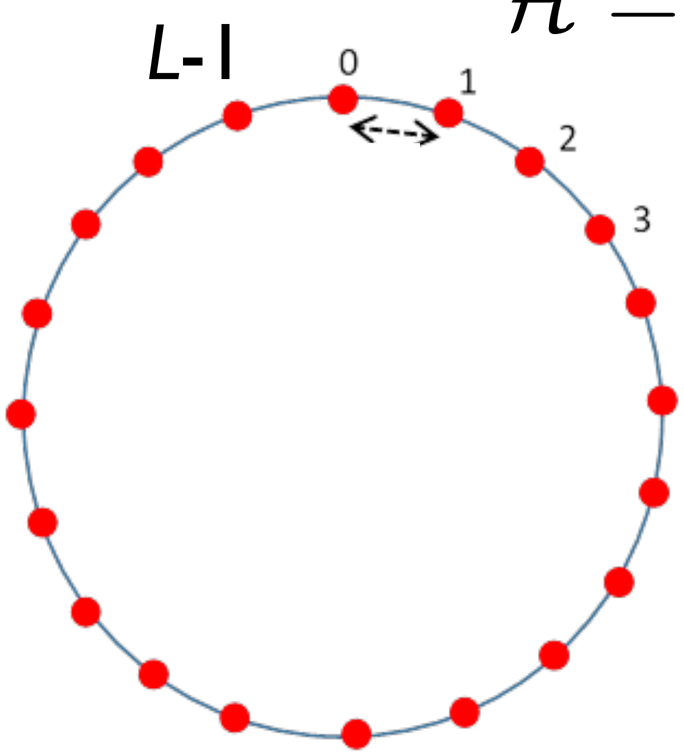
Number of particles: conserved ← U(1) symmetry

Lattice translation symmetry +

spatial inversion or time reversal symmetry

e.g. 1D spinless Hubbard model with periodic b.c. $c_L \equiv c_0$

$$\mathcal{H} = -t \sum_{j=0}^{L-1} \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) + V \sum_{j=0}^{L-1} n_j n_{j+1}$$



Lattice translation $\mathcal{T}$ $\quad \mathcal{T} c_j \mathcal{T}^{-1} = c_{j+1}$

Translation inv. $\quad [\mathcal{T}, \mathcal{H}] = 0$

LSM Variational Argument

Ground state $$\mathcal{H}|\Psi_0\rangle = E_0|\Psi_0\rangle$$

(very complicated — we don't need to know it exactly its EXISTENCE is enough!)

$e^{i\theta N} = e^{i\theta \sum_j n_j}$ global U(1) transformation $c_j \to e^{i\theta} c_j$

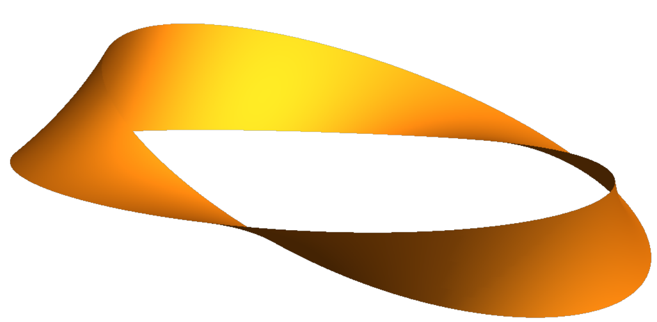
"Slow twist" (NOT symmetry) $$\mathcal{U} \equiv \exp\left(\sum_j \frac{2\pi i j}{L} n_j\right)$$

$$\mathcal{U}^\dagger c_j \mathcal{U} = \exp\left(\frac{2\pi i j}{L}\right) c_j$$

consistent with PBC $c_L \equiv c_0$

$$\mathcal{U}^\dagger c_0 \mathcal{U} = \exp\left(\frac{2\pi i 0}{L}\right) c_0 = c_0$$

$$\mathcal{U}^\dagger c_L \mathcal{U} = \exp\left(\frac{2\pi i L}{L}\right) c_L = c_L$$



LSM Variational Argument

$$\mathcal{U}^\dagger \mathcal{H} \mathcal{U} = -t \sum_{j=0}^{L-1} \left(e^{-2\pi i/L} c_{j+1}^\dagger c_j + e^{2\pi i/L} c_j^\dagger c_{j+1} \right) + V \sum_{j=0}^{L-1} n_j n_{j+1}$$

$$\mathcal{H} = -t \sum_{j=0}^{L-1} \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) + V \sum_{j=0}^{L-1} n_j n_{j+1}$$

$$\mathcal{U}^\dagger \mathcal{H} \mathcal{U} - \mathcal{H} = t \frac{2\pi i}{L} \sum_j \left(c_{j+1}^\dagger c_j - c_j^\dagger c_{j+1} \right)$$

→ expectation value vanishes

$$+ t \left(\frac{2\pi}{L} \right)^2 \sum_j \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) + O(\frac{1}{L^2})$$

$$\langle \Psi_0 | \left(\mathcal{U}^\dagger \mathcal{H} \mathcal{U} - \mathcal{H} \right) | \Psi_0 \rangle = O(\frac{1}{L})$$

$\mathcal{U}|\Psi_0\rangle$ is a low-energy state

Does it mean anything?

$\mathcal{U}|\Psi_0\rangle$ could be (almost) identical to $|\Psi_0\rangle$

Are they different?

"filling factor"
(particle # / site)

$$\mathcal{T}|\Psi_0\rangle = e^{iP_0}|\Psi_0\rangle$$

$$\mathcal{U}^\dagger \mathcal{T} \mathcal{U} = e^{2\pi i \sum_j n_j / L}\mathcal{T} = e^{2\pi \nu i}\mathcal{T}$$

$$\nu = \frac{\sum_j n_j}{L} = \frac{N}{L}$$

$$\mathcal{U} \equiv \exp\left(\sum_j \frac{2\pi i j}{L} n_j\right)$$

$\mathcal{U}|\Psi_0\rangle$ is a low-energy state
different from $|\Psi_0\rangle$
if ν is NOT integer!

$$\mathcal{T}\left(\mathcal{U}|\Psi_0\rangle\right) = e^{iP_0 + 2\pi\nu i}\left(\mathcal{U}|\Psi_0\rangle\right)$$

Statement of LSM theorem

Quantum Many-Body System (in 1D) with
- global U(1) symmetry

AND

- lattice translation symmetry

WITH a fractional (non-integer) filling factor ν

↓

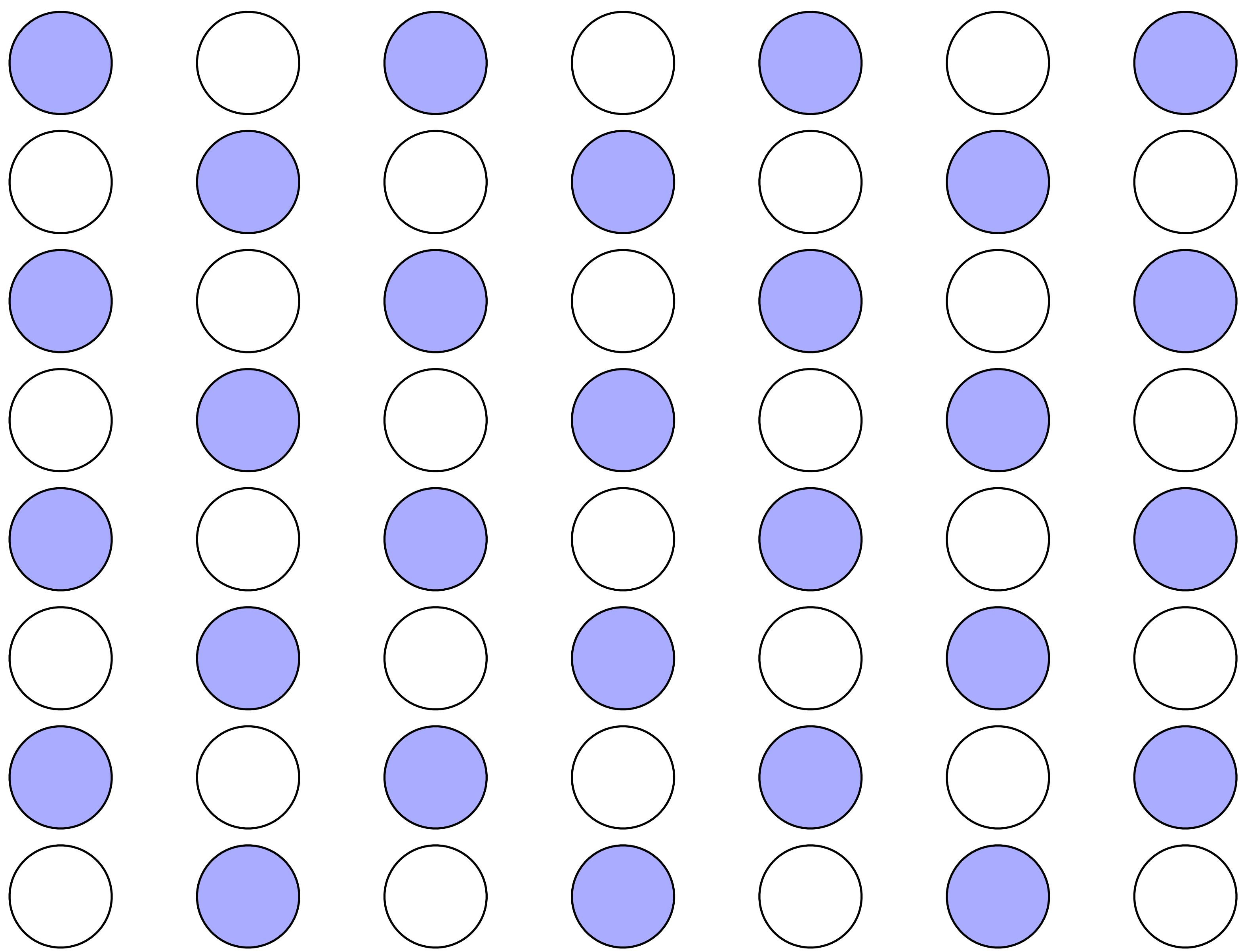
- gapless excitations above the ground state

OR

- multiple, degenerate ground states below gap

- ~~unique ground state below gap~~
~~"featureless (trivial) insulator"~~

Intuitive picture for the LSM theorem: gapped phase needs the particles to be "locked", and the density of the particles must be commensurate with the lattice.



1 particle/
unit cell
(= 2 sites)

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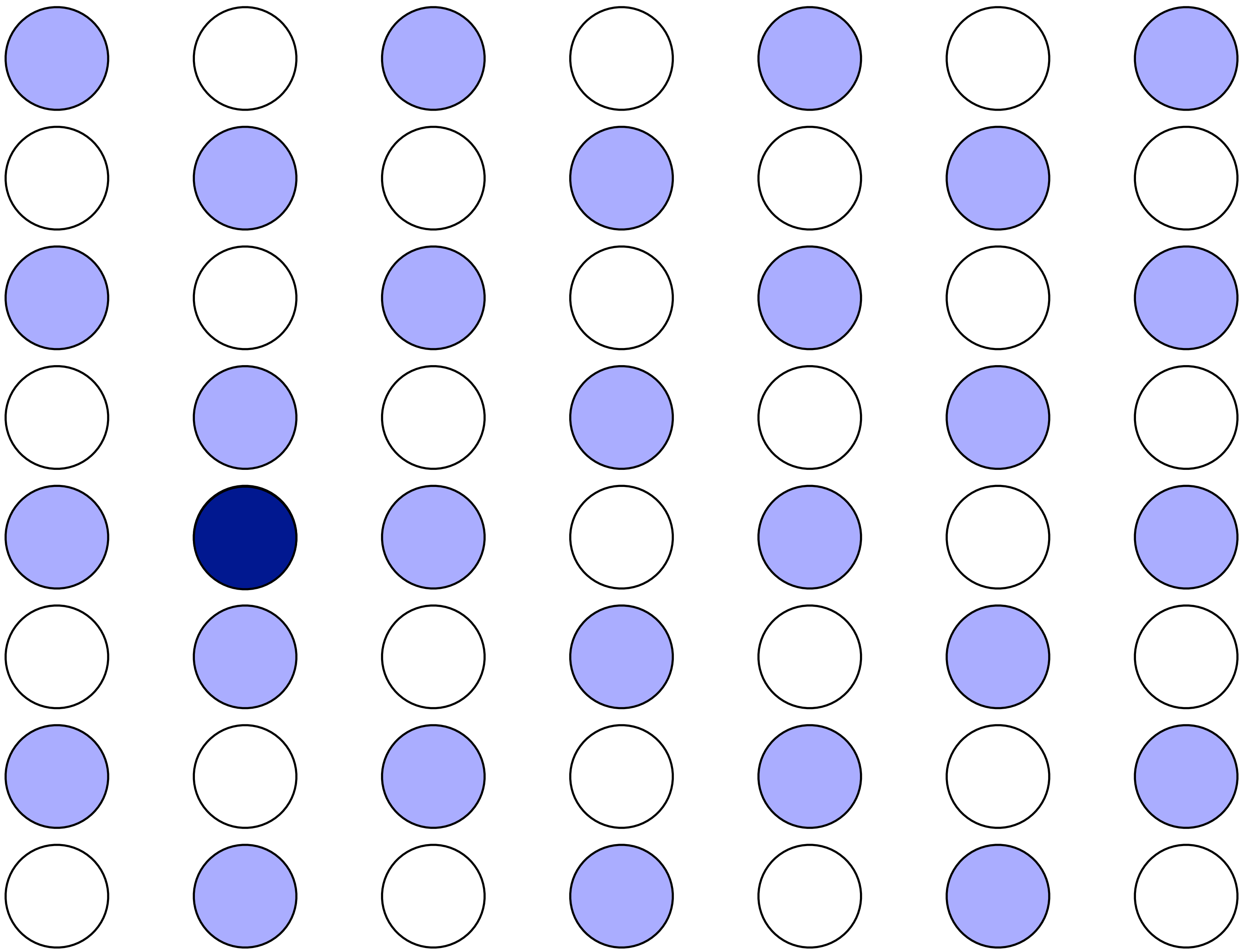
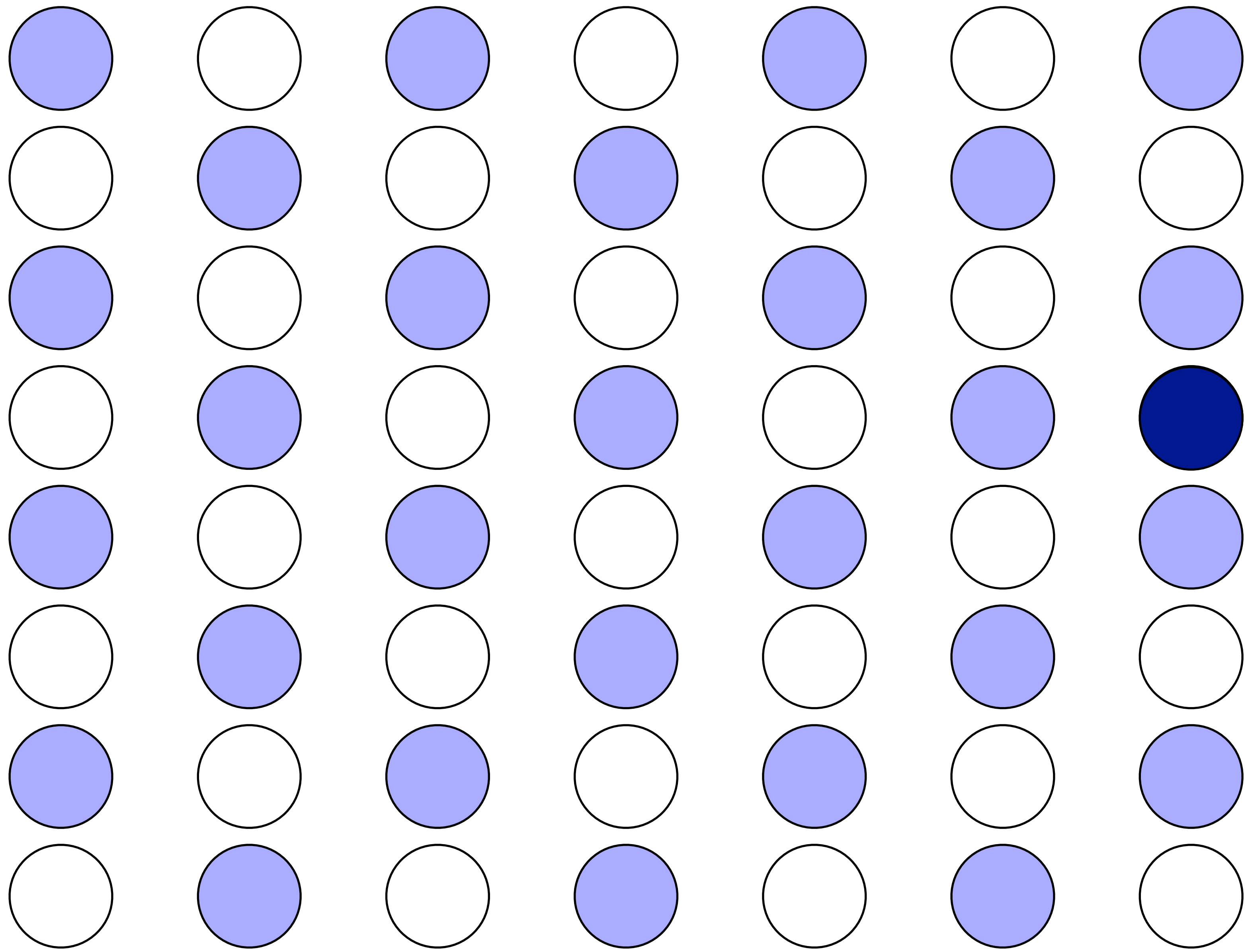


1 particle/
unit cell
(= 2 sites)

add extra
particles
(“doping”)

Intuitive picture for the LSM theorem: gapped phase needs the particles to be "locked", and the density of the particles must be commensurate with the lattice.



1 particle/ unit cell (= 2 sites)

add extra particles ("doping")

mobile carriers

Original LSM paper in 1961

ANNALS OF PHYSICS: **16**, 407–466 (1961)

not about the famous theorem!?

Two Soluble Models of an Antiferromagnetic Chain

ELLIOTT LIEB, THEODORE SCHULTZ, AND DANIEL MATTIS

Thomas J. Watson Research Center, Yorktown, New York

II. THE *XY* MODEL

A. FORMULATION

The first model consists of N spin $\frac{1}{2}$'s (N even) arranged in a row and having only nearest neighbor interactions. It is

$$H_\gamma = \sum_i [(1+\gamma) S^x_i S^x_{i+1} + (1-\gamma) S^y_i S^y_{i+1}], \qquad (2.1)$$

a's and $a^\dagger$'s do not preserve this mixed set of canonical rules. However, it is possible to transform to a new set of variables that are strictly Fermi operators and in terms of which the Hamiltonian is just as simple.[1] Let

$$c_i \equiv \exp\left[\pi i \sum_1^{i-1} a_j^\dagger a_j\right] a_i$$

Main Result:

Exact solution of S=1/2 XY chain by mapping to free fermions (Jordan-Wigner transformation)

Über das Paulische Äquivalenzverbot.

Von **P. Jordan** und **E. Wigner** in Göttingen.

(Eingegangen am 26. Januar 1928.)

Die Arbeit enthält eine Fortsetzung der kürzlich von einem der Verfasser vorgelegten Note „Zur Quantenmechanik der Gasentartung", deren Ergebnisse hier wesentlich erweitert werden. Es handelt sich darum, ein ideales oder nichtideales, dem Paulischen Äquivalenzverbot unterworfenes Gas zu beschreiben mit Begriffen, die keinen Bezug nehmen auf den abstrakten Koordinatenraum der Atomgesamtheit des Gases, sondern nur den gewöhnlichen dreidimensionalen Raum benutzen. Das

wird ern
dreidime
plikation
kularer (
antwortli
entsprech

wenn wir die Größen a, $a^\dagger$ durch

$$\left.\begin{aligned} a_p(q') &= v(q') \cdot b_p(q'), \\ a_p^\dagger(q') &= b_p^\dagger(q') \cdot v(q'); \end{aligned}\right\} \tag{31}$$

$$v(q') = \prod_{q'' \leqq q'} \{1 - 2\,N(q'')\} \tag{32}$$

definieren. Hier ist also $v(q')$ das Produkt der Größen $1 - 2\,N(q'')$ für $q'' = q'$ und alle v o r q' kommenden q''. Es ist also $v(q')$ eine Diagonalmatrix, deren Diagonalelemente sämtlich gleich $+1$ oder -1 sind; und es wird

$$[v(q')]^2 = 1. \tag{33}$$

Where was the LSM theorem??

Apparently, the authors themselves did not think the theorem was too important.
They proved the theorem just for S=1/2 chain at zero magnetic field…

Perhaps the theorem had drawn little attention for more than 20 years after its birth in 1961

Where was the LSM theorem??

Appendix….

APPENDIX B. NONDEGENERACY OF THE GROUND STATE AND ABSENCE OF AN ENERGY GAP IN THE HEISENBERG MODEL

We prove two exact theorems about the ground state and excitation spectrum for a Heisenberg model with nearest neighbor interactions in one dimension.

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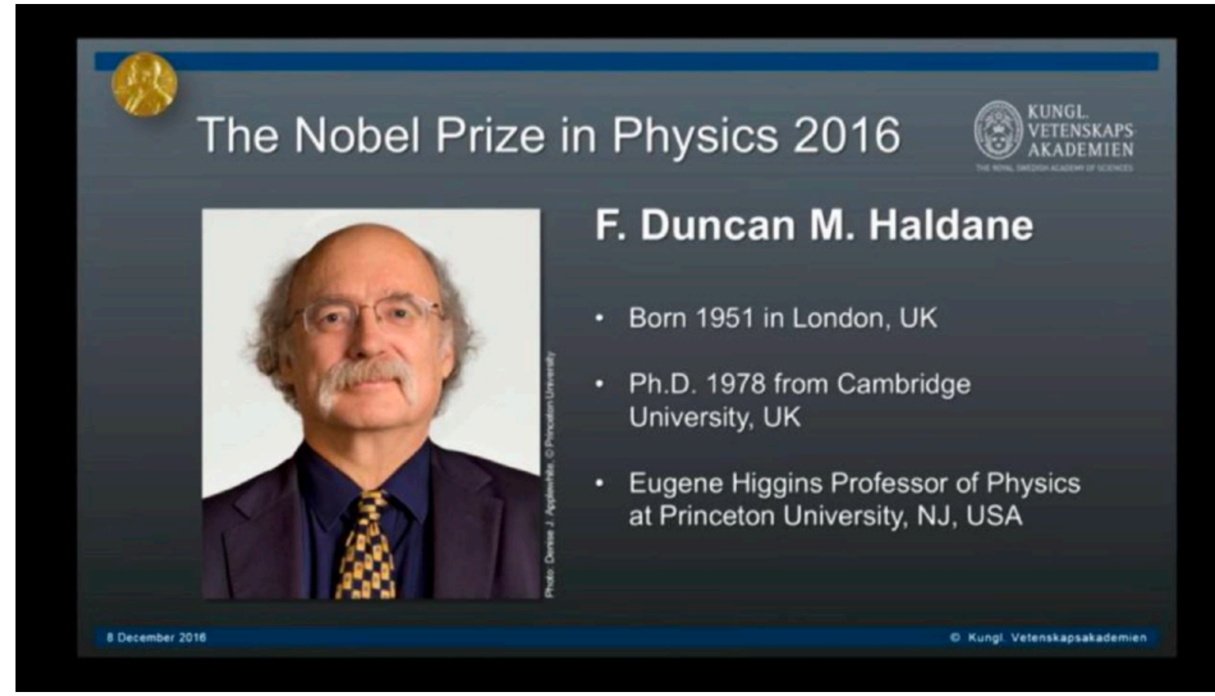
Haldane "Conjecture" in 1981

S=1/2, 3/2, 5/2, ...

Gapless "Quantum Critical"

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle \propto \left(\frac{1}{r} \right)^{\eta}$$



S=1, 2, 3.....

Non-vanishing excitation gap ("Haldane gap")

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle \propto \exp \left(-\frac{r}{\xi} \right)$$

Against the "common sense" at the time ⇒ "conjecture"

Affleck-Lieb 1986

Generalization of the original LSM theorem for S=1/2 to arbitrary spin quantum number S

Letters in Mathematical Physics **12** (1986) 57–69. 57
© 1986 by *D. Reidel Publishing Company.*

A Proof of Part of Haldane's Conjecture on Spin Chains

IAN AFFLECK* and ELLIOTT H. LIEB**
Departments of Mathematics and Physics, Princeton University, P.O. Box 708, Princeton, NJ 08544, U.S.A.

(Received: 10 March 1986)

Abstract. It has been argued that the spectra of infinite length, translation and $U(1)$ invariant, anisotropic, antiferromagnetic spin s chains differ according to whether s is integral or $\frac{1}{2}$ integral: There is a range of parameters for which there is a unique ground state with a gap above it in the integral case, but no such range exists for the $\frac{1}{2}$ integral case. We prove the above statement for $\frac{1}{2}$ integral spin. We also prove that for *all s*, *finite length* chains have a unique ground state for a wide range of parameters. The argument was extended to $SU(n)$ chains, and we prove analogous results in that case as well.

S: half-odd-integer
→ gapless or 2-fold g.s. degeneracy

integer S : no constraint from LSM
→ may have a unique gapped ground state consistent with Haldane conjecture!

Spin System as Many Particles

Spin S: $S^z = -S, -S+1, \ldots, S-1, S$

M.O.-Yamanaka-Affleck 1997

Identify, say, $S^z = -S$ state as "vacuum"

increase S^z by 1 ⇔ add a particle (magnon)

$$S_j^z = -S + n_j$$

magnetization per site

$$m = \langle S_j^z \rangle = -S + \langle n_j \rangle = -S + \nu$$

zero magnetization (ground state of antiferromagnet)

$$m = 0 \qquad \nu = S$$

fractional filling if and only if S is half-odd-int

Why Haldane Gap?

Standard(?) view:

topological term of the O(3) non-linear sigma model present only for half-odd-integer spin *S*

Intuitive(?) view:

half-odd-integer spin *S*: fractional (1/2+integer) filling

integer spin *S*: integer filling → can be "trivial" insulator

naturally obtained by generalizing the LSM theorem to many particle systems [Yamanaka-MO-Affleck 1997]

$$m = \langle S_j^z \rangle = -S + \langle n_j \rangle = -S + \nu$$

zero magnetization (ground state of antiferromagnet)

$$m = 0 \qquad \nu = S$$

Why not in LSM?

APPENDIX B. NONDEGENERACY OF THE GROUND STATE AND ABSENCE OF AN ENERGY GAP IN THE HEISENBERG MODEL

We prove two exact theorems about the ground state and excitation spectrum for a Heisenberg model with nearest neighbor interactions in one dimension. The generalization to longer range interactions and higher-dimensional lattices is indicated. A further generalization to particles of spin $\neq \frac{1}{2}$ and a discussion of the ordering of excited state energy levels has been submitted for publication in the *Journal of Mathematical Physics* by Lieb and Mattis.

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Perhaps refers to this paper

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 4 JULY-AUGUST 1962

Ordering Energy Levels of Interacting Spin Systems

ELLIOTT LIEB AND DANIEL MATTIS
Thomas J. Watson Research Center, International Business Machines Corporation, Yorktown Heights, New York
(Received October 6, 1961)

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But no mention is actually made on the generalization of LSM theorem?!

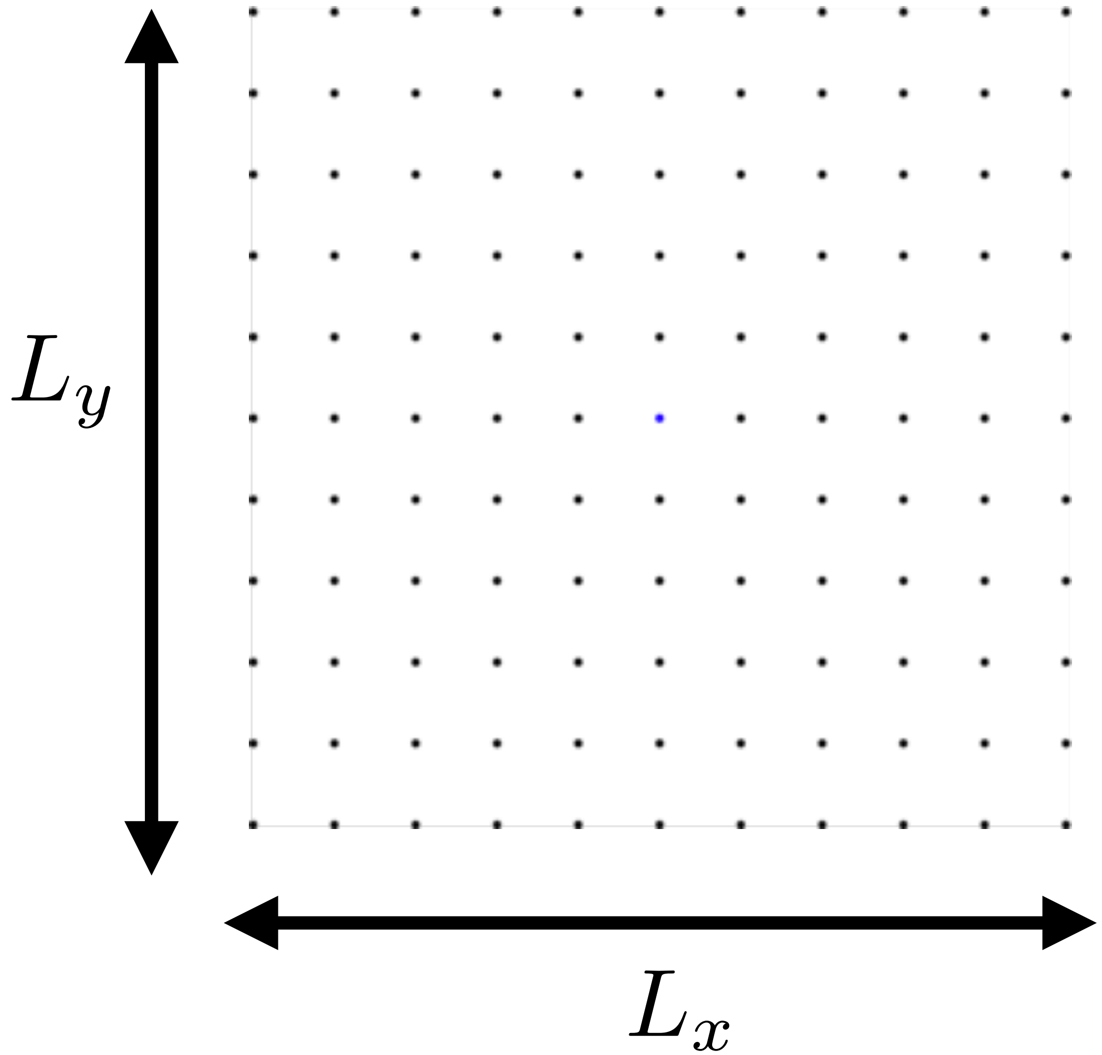
Maybe….

LSM in 1961 thought they can generalize their theorem to general *S*,
but then realized the proof "fails" for integer *S*

So they scrapped the generalization
(until Affleck-Lieb paper in 1985,
but after Haldane conjecture)

…. perhaps they just missed the clue of
the "Haldane gap"??

Higher Dimensions?



LSM twist in *x* direction

$$\mathcal{U}_x = \exp\left(\frac{2\pi i}{L_x}\sum_{\vec{r}} x n_{\vec{r}}\right)$$

$$\vec{r} = (x, y) \in \mathbb{Z}^2$$

Energy gain due to the twist $O(\frac{1}{{L_x}^2}) \times L_x L_y = O(\frac{L_y}{L_x})$

Not small….?!

Anisotropic Limit

LSM variational argument works, if $L_y/L_x \to 0$

while $L_x, L_y \to \infty$, as already pointed out in LSM(1961)

In two dimensions we consider a square lattice of N sites in the x-direction and of $M = O(N^\nu)$ sites in the y-direction, where $0 < \nu < 1$. The Hamiltonian is assumed cyclic in the sense that

$$\mathbf{S}_{n,\,M+1} = \mathbf{S}_{n,\,1} \tag{B-25a}$$

and

$$\mathbf{S}_{N+1,\,m} = \mathbf{S}_{1,\,m}\,, \tag{B-26}$$

i.e., the lattice is wrapped on a torus. We take for the operator $\mathcal{O}^k$,

$$\mathcal{O}^k = \exp\left(ik \sum_{n=1}^{N} \sum_{m=1}^{M} nS^z_{n,m}\right). \tag{B-27}$$

This operator twists the direction of all spins with the same x-coordinate by the same amount. Ψ_k is constructed and its orthogonality to the ground state is proved precisely as in one dimension. Instead of (B-24), one now has

$$\langle \Psi_k \mid H \mid \Psi_k \rangle \leqq E_0 + (2\pi^2/N^{1-\nu}); \tag{B-28}$$

so again there is no energy gap. Because the excitation energy of exact low-lying states should not depend on the shape of the entire lattice, there should be no energy gap for a lattice of $N \times N$ sites either. The particular state Ψ_k is unfortunately not sufficiently like an exact low-lying excited state to give this result.

A similar extension to three dimensions is obvious.

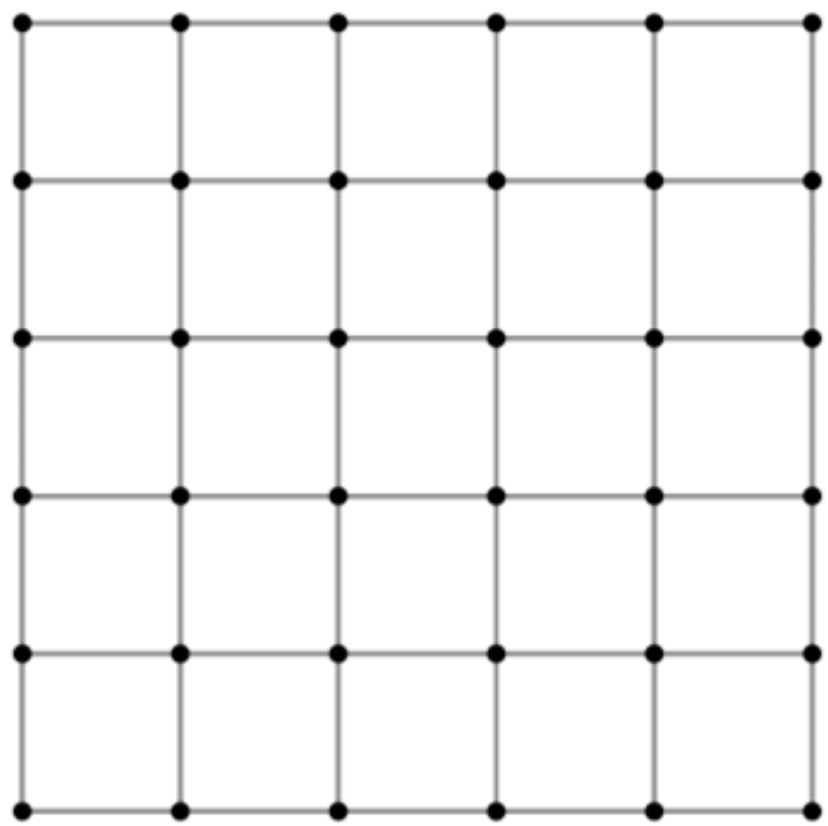
But is this really 2D limit?

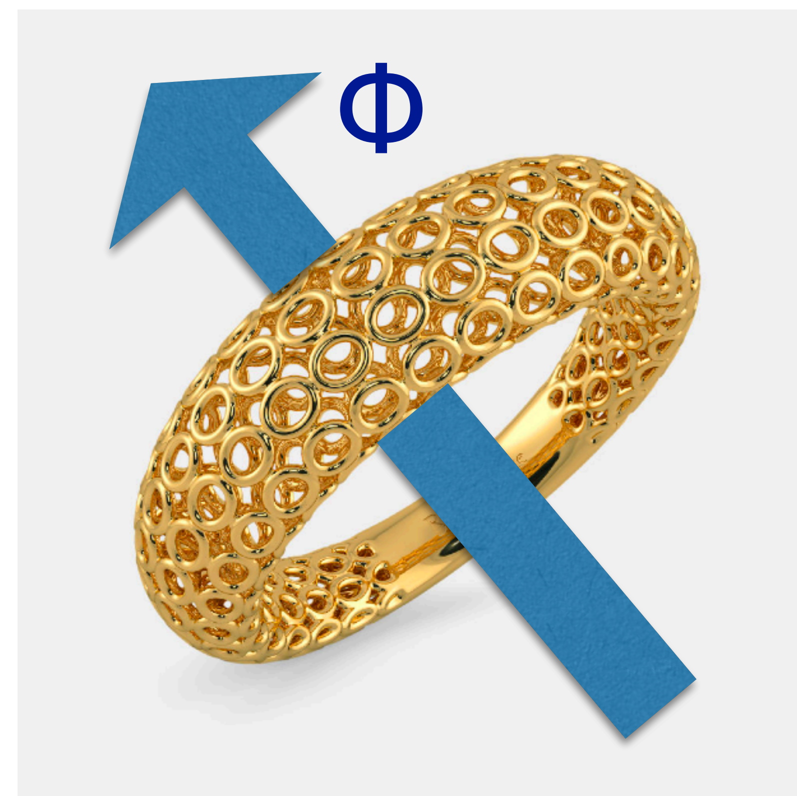
Can we show LSM for isotropic 2D limit?

Many Particles on Periodic Lattice

For example, consider a many-particle system on the square lattice of $L_x \times L_y$ with periodic boundary conditions

assume particle number conservation (U(1) symmetry)





assume that the system is gapped, and consider the adiabatic insertion of unit flux quantum through the "hole"

M. O. 2000

Adiabatic Flux Insertion

(i) Increase Aharonov-Bohm flux Φ adiabatically from 0 to Φ_0(=2π) $|\Psi_0\rangle \to |\Psi_0'\rangle$

Hamiltonian for the final state is different from the original one, but we can

(ii) eliminate the unit flux quantum by the large gauge transformation

$$U_x \mathcal{H}(\Phi = 2\pi) U_x{}^{-1} = \mathcal{H}(\Phi = 0)$$

$$U_x = \exp\left(\frac{2\pi i}{L_x} \sum_{\vec{r}} x n_{\vec{r}}\right)$$

$$|\Psi_0\rangle \to |\Psi_0'\rangle \to U_x |\Psi_0'\rangle$$

variational argument replaced by adiabatic process

Large Gauge Transformation

Initial Groundstate $|\Psi_0\rangle$

$$T_x|\Psi_0\rangle = e^{iP_x^{(0)}}|\Psi_0\rangle$$

groundstate of $\mathcal{H}(0)$

Final State $|\Psi_0'\rangle = \mathcal{F}_x|\Psi_0\rangle$

$$T_x|\Psi_0'\rangle = e^{iP_x^{(0)}}|\Psi_0'\rangle$$

groundstate of $\mathcal{H}(2\pi)$

Large gauge transformation

$|\tilde{\Psi}_0'\rangle \equiv U_x|\Psi_0'\rangle$ must be a groundstate of $\mathcal{H}(0)$

$$U_x = \exp\left(\frac{2\pi i}{L_x}\sum_{\vec{r}} x n_{\vec{r}}\right) \qquad {U_x}^{-1} T_x U_x = T_x \exp\left(\frac{2\pi i}{L_x}\sum_{\vec{r}} n_{\vec{r}}\right)$$

$$T_x|\tilde{\Psi}_0'\rangle = e^{i\left(P_x^{(0)} + \frac{2\pi}{L_x}\sum_{\vec{r}} n_{\vec{r}}\right)}|\tilde{\Psi}_0'\rangle$$

Momentum Shift

$$P_x^{(0)} \to P_x^{(0)} + \frac{2\pi}{L_x} \sum_{\vec{r}} n_{\vec{r}}$$

total number of particles (conserved)

We are usually interested in the thermodynamic limit for a fixed particle density (particle # / unit cell) ν

Suppose $\nu = \frac{p}{q}$ and choose L_y to be a coprime with q

$$\Delta P_x = \frac{2\pi}{L_x} L_x L_y \nu = 2\pi L_y \frac{p}{q}$$

Lattice momentum is defined modulo 2π momentum shifted if $q \neq 1$ (fractional filling)

The final state is different from the initial ground state

⇒ **ground-state degeneracy!**

LSM in arbitrary dimensions

LSM 1961, Affleck-Lieb 1985, M.O.-Yamanaka-Affleck 1997, M. O. 2000, Hastings 2004,…

Periodic (translation invariant) lattice ⇒ unit cell

U(1) symmetry ⇒ conserved particle number

ν : number of particle per unit cell (filling fraction)

$\nu = p/q \quad \Rightarrow$

"ingappability"

- system is gapless

must be in a nontrivial phase!

OR

- gapped with q-fold degenerate ground states

~~gapped with unique ground state~~

Recent Developments

nature physics
ARTICLES
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Topological order and absence of band insulators at integer filling in non-symmorphic crystals

Siddharth A. Parameswaran[1], Ari M. Turner[2], Daniel P. Arovas[3] and Ashvin Vishwanath[1,4]*

Non-symmorphic lattice with “glide symmetry”:
“effective unit cell” is half of the unit cell

$$\nu_{\text{eff}} = \frac{\nu}{2}$$

LSMOH-type restriction
even when $\nu \in \mathbb{Z}$

Crystallographic Symmetries

Filling constraints for spin-orbit coupled insulators in symmorphic and nonsymmorphic crystals

Haruki Watanabe[a], Hoi Chun Po[b], Ashvin Vishwanath[b,c], and Michael Zaletel[d,1]

PNAS | November 24, 2015 | vol. 112 | no. 47 | 14551–14556

Table 1. Summary of ν_{min} for elementary space groups

ITC no.	Key elements	Minimal filling: AI*	Ent†	Bbb‡	Manifold name
1	(Translation)	2	2	2	Torus
4	2_1	4	4	4	Dicosm
144/145	$3_1/3_2$	6	6	6	Tricosm
76/78	$4_1/4_3$	8	8	8	Tetracosm
77	4_2	4	4	4	
80	4_1	4	4	4	
169/170	$6_1/6_5$	12	12	12	Hexacosm
171/172	$6_2/6_4$	6	6	6	
173	6_3	4	4	4	
19	2_1, 2_1	8	4	8	Didicosm
24	2_1, 2_1	4	2	4	
7	Glide	4	4	4	First amphicosm
9	Glide	4	4	4	Second amphicosm
29	Glide, 2_1	8	4	8	First amphidicosm
33	Glide, 2_1	8	4	8	Second amphidicosm

*The minimal filling required to form a symmetric atomic insulator.
†ν_{min} obtained in *Extension to 3D Symmorphic and Nonsymmorphic Crystals*. Bounds are not tight for nos. 19, 24, 29, and 33.
‡ν_{min} obtained in *Alternative Method: Putting Sym-SRE Insulators on Bieberbach Manifolds*. All bounds are tight.

LSM for Discrete Symmetry?

Proofs/arguments for the original LSM do not work

1D LSM

$$U = \exp\left(\frac{2\pi i}{L} \sum_j j S^z_j \right)$$

Lieb-Schultz-Mattis
1961

variational low-energy state

2D and higher

Adiabatic insertion of magnetic flux
(U(1) gauge field)

M.O. 2000
Hastings 2004〜

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LSM for Discrete Symmetry

[1D]

MPS-based "proof" Chen-Gu-Wen 2011

Field-theory argument Fuji 2014

Mathematical proof Ogata-Tachikawa-Tasaki 2020

[2D and higher]

Many statements for space group symmetries etc.

Po-Watanabe-Jian-Zalatel 2017, Else-Thorngren 2020

But the argument is either at abstract level, or relying on

Schmidt decomposition (OK for fixed width but…)

or "trivial" degeneracy of odd-site sytems

Watanabe-Po-Vishwanath-Zalatel 2015

So we will try to give a convincing physics argument…

Rigorous Proof for 1D in 2020

Cornell University

We g
the Simons F

arXiv.org > math-ph > arXiv:2004.06458

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Mathematical Physics

[Submitted on 14 Apr 2020 (v1), last revised 26 Apr 2020 (this version, v2)]

General Lieb-Schultz-Mattis type theorems for quantum spin chains

Yoshiko Ogata, Yuji Tachikawa, Hal Tasaki

We develop a general operator algebraic method which focuses on projective representations of symmetry group for proving Lieb-Schultz-Mattis type theorems, i.e., no-go theorems that rule out the existence of a unique gapped ground state (or, more generally, a pure split state), for quantum spin chains with on-site symmetry. We first prove a theorem for translation invariant spin chains that unifies and extends two theorems proved by two of the authors in [OT1]. We then prove a Lieb-Schultz-Mattis type theorem for spin chains that are invariant under the reflection about the origin and not necessarily translation invariant.

Comments:	22 pages; v2: typos corrected and references added; the reference [OT1] in the abstract refers to arXiv:1808.08740
Subjects:	**Mathematical Physics (math-ph)**; Strongly Correlated Electrons (cond-mat.str-el); Operator Algebras (math.OA)
Report number:	IPMU-20-0033
Cite as:	arXiv:2004.06458 **[math-ph]**
	(or arXiv:2004.06458v2 **[math-ph]** for this version)

Example: XYZ model

Yuan Yao (ISSP→RIKEN) & M.O.
arXiv:2010.09244

"XYZ" spin model on the square lattice of size $L_1 \times L_2$

$$\mathcal{H} = \sum_{\langle \vec{r}, \vec{r}' \rangle} \left(J_X S^x_{\vec{r}} S^x_{\vec{r}'} + J_Y S^y_{\vec{r}} S^y_{\vec{r}'} + J_Z S^z_{\vec{r}} S^z_{\vec{r}'} \right)$$

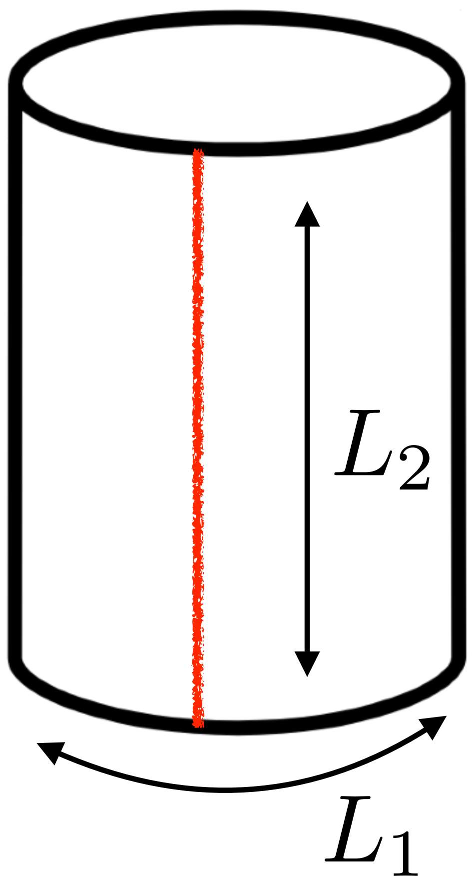
On-site discrete symmetry of $Z_2 \times Z_2$ (dihedral sym.)
(π-rotation of spins about *x*, *y*, and *z* axes)
Lattice translation symmetry T_1, T_2

We can "twist" the boundary condition along *x*-direction by π-rotation about *z*-axis

Twisted Boundary Condition

$$\mathcal{H}^{\text{twist}} = \sum_{\langle \vec{r},\vec{r}' \rangle \notin \text{seam}} \left(J_X S^x_{\vec{r}} S^x_{\vec{r}'} + J_Y S^y_{\vec{r}} S^y_{\vec{r}'} + J_Z S^z_{\vec{r}} S^z_{\vec{r}'} \right)$$

$$+ \sum_{\langle \vec{r},\vec{r}' \rangle \in \text{seam}} \left(-J_X S^x_{\vec{r}} S^x_{\vec{r}'} - J_Y S^y_{\vec{r}} S^y_{\vec{r}'} + J_Z S^z_{\vec{r}} S^z_{\vec{r}'} \right)$$



$$[T_1, \mathcal{H}^{\text{twist}}] \neq 0 \quad \textbf{but} \quad [\tilde{T}_1, \mathcal{H}^{\text{twist}}] = 0$$

$$\tilde{T}_1 \equiv T_1 \prod_{\vec{r} \in \text{seam}} e^{i\pi S^z_{\vec{r}}}$$

translation
+ discrete gauge tr.

Global π-rotation of spins about x-axis

$$R^\pi_x \equiv \prod_{\vec{r}} e^{i\pi S^x_{\vec{r}}} \qquad [R^\pi_x, \mathcal{H}^{\text{twist}}] = 0$$

Two Symmetries under the Twisted BC

$$\tilde{T}_1 \equiv T_1 \prod_{\vec{r} \in \mathrm{seam}} e^{i\pi S^z_{\vec{r}}} \qquad R^\pi_x \equiv \prod_{\vec{r}} e^{i\pi S^x_{\vec{r}}}$$

$$\tilde{T}_1 R^\pi_x = R^\pi_x \tilde{T}_1 (-1)^{2SL_2} \Leftarrow e^{i\pi S^z_{\vec{r}}} e^{i\pi S^x_{\vec{r}}} = (-1)^{2S} e^{i\pi S^x_{\vec{r}}} e^{i\pi S^z_{\vec{r}}}$$

The two symmetry operators ***anticommute***
if *S* is half-odd-integer and L_2 is chosen to be odd

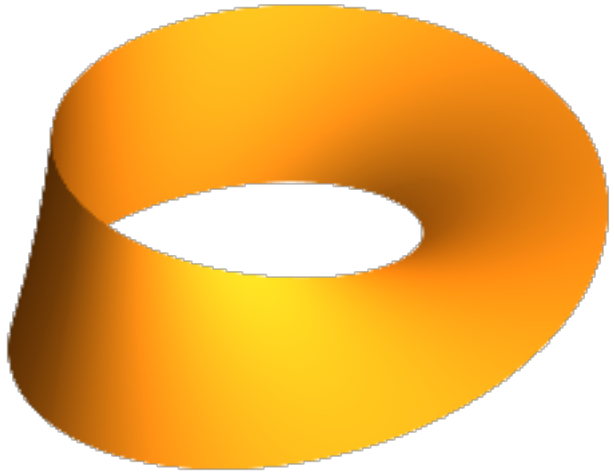
⇒ Ground states (and all the eigenstates) of the Hamiltonian under the twisted b.c. are **exactly two-fold degenerate!**

Hirano-Katsura-Hatsugai 2008
for U(1) symmetric systems

What Does This Mean?

This argument does not apply directly to periodic b.c.
If the degeneracy is only an artifact of the twisted b.c. it would not mean much.

But we argue that **the degeneracy is "robust" and present also for the periodic b.c.**

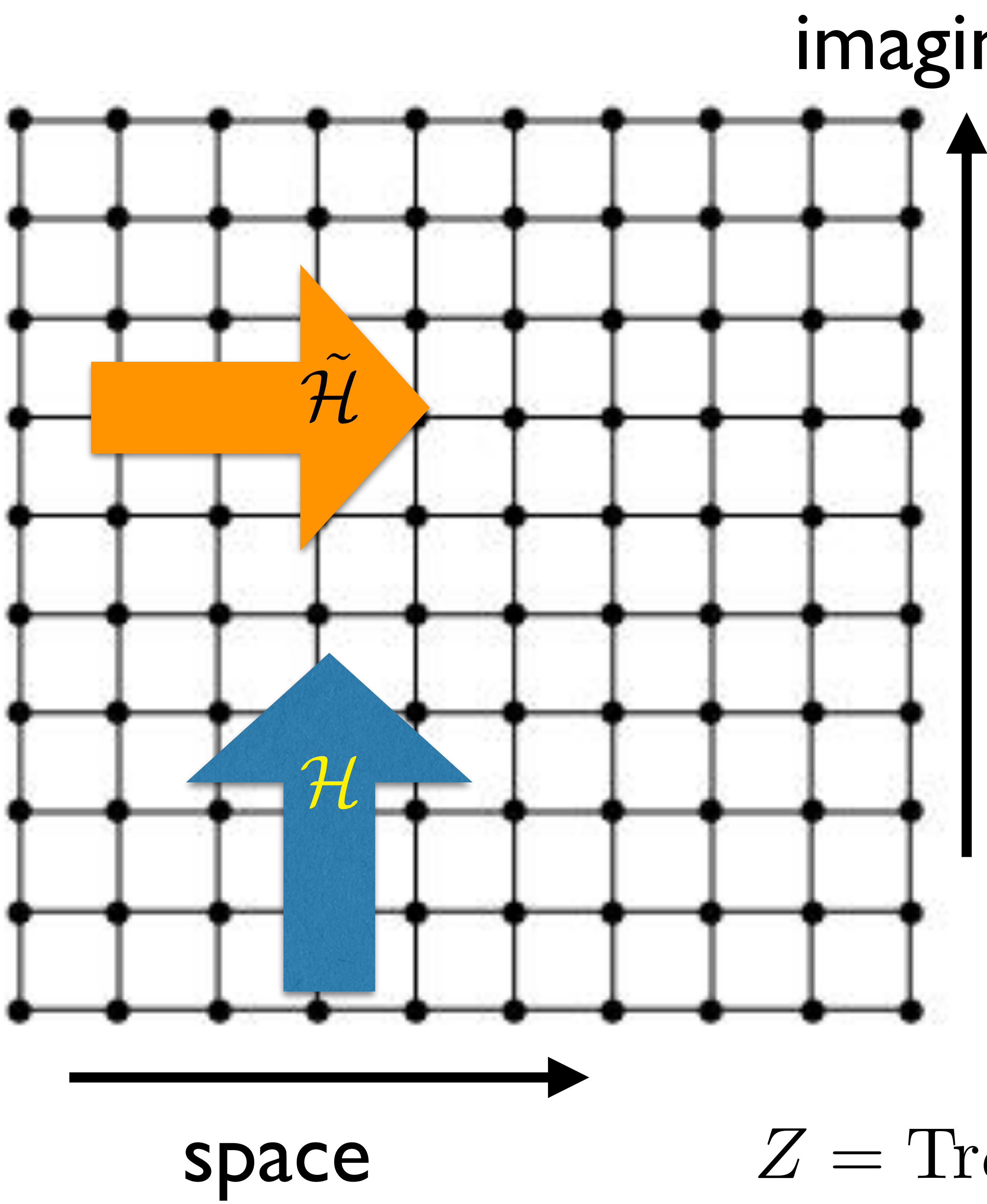


Physical (Hand-Waving) Argument

If the ground state is gapped and unique under the periodic b.c., the system should not have any order (conventional or topological). The absence of the order implies that the system should be insensitive to the twist of the b.c. (in a large enough system)

Therefore, the exact ground-state degeneracy under the twisted b.c. does imply some order (conventional or topological), and the (quasi) ground-state degeneracy under the periodic b.c.

More Formal (Less Hand-Waving?) Approach



"Quantum Transfer Matrix" along spatial direction

Betsuyaku 1984

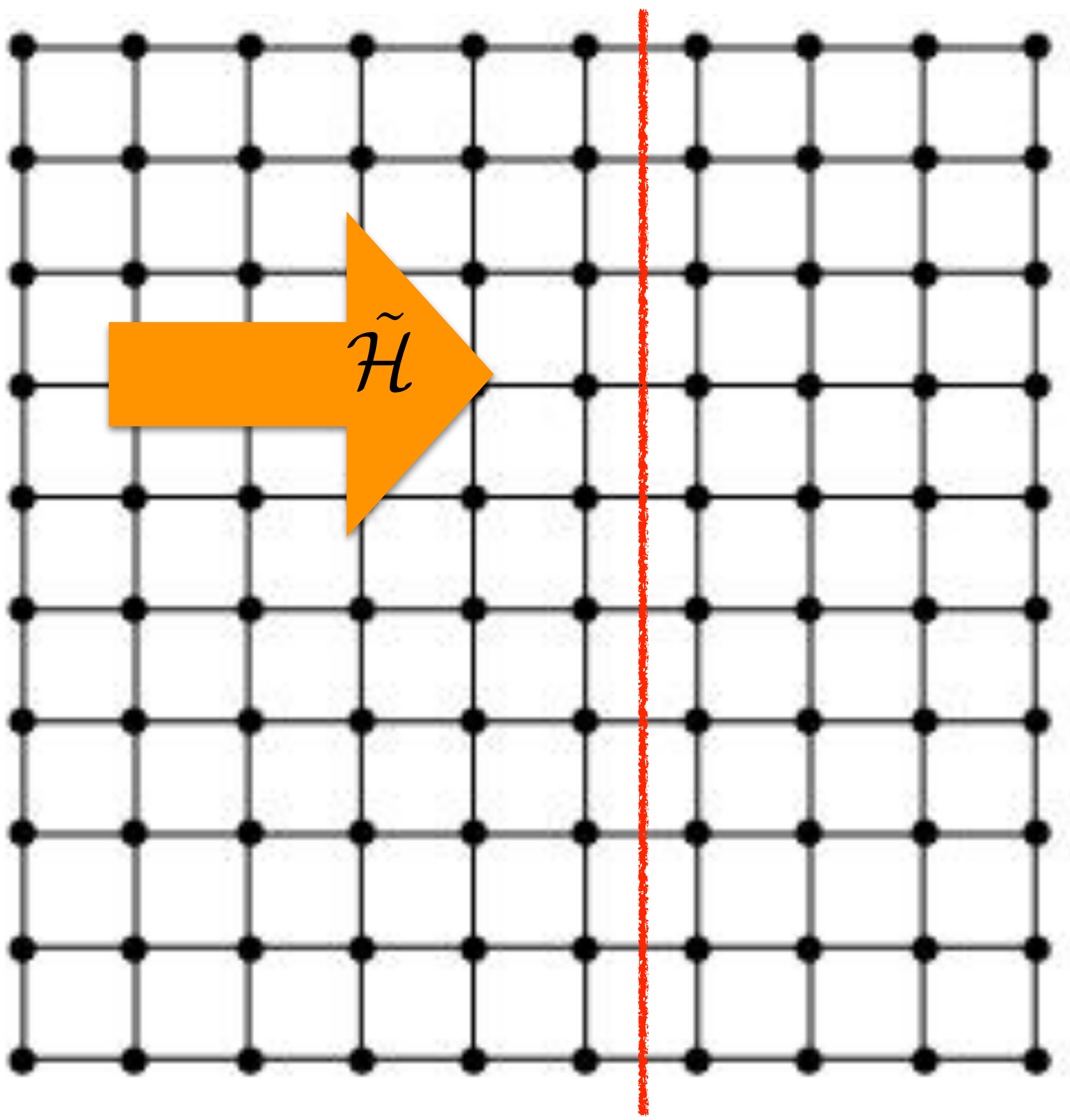
$$\mathcal{T} = e^{-\tilde{\mathcal{H}}}$$

$$Z = \mathrm{Tr} e^{-L_1 \tilde{\mathcal{H}}}$$

Trotter-Suzuki decomposition
→ path integral formulation

$$Z = \mathrm{Tr} e^{-\beta \mathcal{H}} \sim \mathrm{Tr} \left(e^{-\beta \mathcal{H}_A / N} e^{-\beta \mathcal{H}_B / N} \right)^N$$

Twisted BC and QTM



$$Z^{\text{twist}} = \mathrm{Tr}\left(\tilde{R}_z^{\pi} e^{-L_1 \tilde{\mathcal{H}}}\right)$$

$\tilde{R}_z^{\pi}$ global symmetry operator
(π-rotation about z)
unitary

cf.) “topological defect line” in CFT

Proof* by Contradiction

Suppose that the ground-state of the original Hamiltonian is gapped and unique under the periodic b.c.

→ the "ground state" $|\tilde{\Psi}_0\rangle$ of the QTM Hamiltonian $\tilde{\mathcal{H}}$ must be also unique

(cf. zero-temperature entropy in the thermodynamic limit)

Symmetry of $\tilde{\mathcal{H}}$

→ the "ground state" is also an eigenstate of $\tilde{R}_z^\pi$

$$\tilde{R}_z^\pi |\tilde{\Psi}_0\rangle = \zeta_0 |\tilde{\Psi}_0\rangle \qquad |\zeta_0| = 1$$

$$Z^{\text{twist}} \sim \langle \tilde{\Psi}_0 | \tilde{R}_z^\pi e^{-L_1 \tilde{\mathcal{H}}} | \tilde{\Psi}_0 \rangle = \zeta_0 Z^{\text{PBC}} \qquad Z^{\text{twist}} \in \mathbb{R}^+$$

up to exponentially small corrections $\qquad \zeta_0 = 1$

Proof* by Contradiction

$$Z^{\mathrm{twist}} \sim \langle \tilde{\Psi}_0 | \tilde{R}_z^{\pi} e^{-L_1 \tilde{\mathcal{H}}} | \tilde{\Psi}_0 \rangle = \zeta_0 Z^{\mathrm{PBC}} \qquad \zeta_0 = 1$$

$$Z^{\mathrm{twist}} \sim Z^{\mathrm{PBC}}$$

up to exponentially small corrections

Then the zero-temperature entropy in the thermodynamic limit must be zero under the twisted b.c.

→ $\mathcal{H}^{\mathrm{twist}}$ must have a unique ground state

Contradiction with the exact ground-state degeneracy of $\mathcal{H}^{\mathrm{twist}}$

→ assumption (unique gapped g.s. of $\mathcal{H}$) was wrong

→ $\mathcal{H}$ with the periodic b.c. must have degenerate ground states!

*: not really rigorous

Thermodynamic Limit

Ruelle "Statistical Mechanics: rigorous results" 1988

$$Z_\Lambda(\Phi) = \mathrm{Tr}_{\mathscr{H}(\Lambda)} \exp[-H(\Lambda)]$$

so that $\Xi(\Lambda, \beta) = Z_\Lambda(\beta\Phi)$, and we define

$$P_\Lambda(\Phi) = N(\Lambda)^{-1} \log Z_\Lambda(\Phi)$$

2.3.3 THEOREM.[9] *If* $\Phi \in \mathscr{B}$, *the following limit exists and is finite*

$$P(\Phi) = \lim_{\Lambda \to \infty} P_\Lambda(\Phi)$$

Free energy density in the thermodynamic limit
at a fixed temperature

Thermodynamic Limit

zero-temperature entropy given by g.s. degeneracy *d*

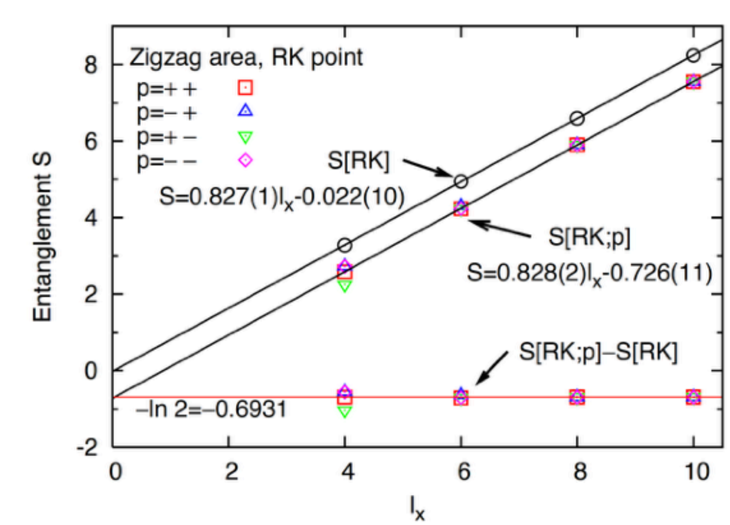
But we need

$$\log Z(\beta, L) \sim \log d + \epsilon_0 \beta L + O(\text{exponentially small})$$

the O(1) quantity log *d* to be well-defined in the limit $\beta \sim L \to \infty$

cf.) "topological entanglement entropy"
Kitaev-Preskill / Levin-Wen

$$S_L \longrightarrow \alpha L - \gamma + \mathcal{O}(L^{-\nu}) \,, \qquad \nu > 0$$



Furukawa-Misguich 2007

What We Have Shown

As a simple example, consider "XYZ" spin model on the square lattice of the size $L_1 \times L_2$

$$\mathcal{H} = \sum_{\langle \vec{r}, \vec{r}' \rangle} \left(J_X S^x_{\vec{r}} S^x_{\vec{r}'} + J_Y S^y_{\vec{r}} S^y_{\vec{r}'} + J_Z S^z_{\vec{r}} S^z_{\vec{r}'} \right)$$

On-site discrete symmetry of $Z_2 \times Z_2$ (π-rotation of spins about x, y, and z axes)

Lattice translation symmetry T_1, T_2

$\Rightarrow$ for half-odd-integer spin (and L_2 is odd),

if the system is gapped, the ground-state must be degenerate under the periodic b.c.

implying (conventional or topological) order

Generalizations and Limitations

Similar constraint if the "spins" within the unit cell transforms a projective representation of the symmetry

SU(N) symmetry etc.

We can often obtain the ground state degeneracy > 2 for the twisted b.c., but at present we cannot deduce the number of ground states for the periodic b.c. (other than it must be > 1). Maybe the number of the degenerate ground states under the twisted b.c. is also robust??

Summary

- Lieb-Schultz-Mattis theorem is one of the few very general yet powerful constraints on quantum many-body systems
- Started as a humble result in an Appendix and had been overlooked for many years, its generality has been gradually appreciated
- Active topic of research for generalization, rigorous proof, etc. (related to anomaly in field theory, topological phases,….) also in recent years