Harvard CMSA
Quantum Matter Seminar
Aug 19 (Thu) 8:30pm- (EDT)

Conformal Field Theory and Modern Numerical Approach to Condensed Matter Physics

Masaki Oshikawa
ISSP, University of Tokyo

Conformal Field Theory

Scale Invariance at the Critical Point

Locality of the Hamiltonian/Action

$\Downarrow$

Invariance under conformal transformation

(locally different scale transformation)

1+1 Dimension:

conformal transformation ⇔ complex analytic function

∞-dimensional "symmetry" (Virasoro Algebra)

What can we deduce from the ∞-dimensional symmetry?

"Conformal Field Theory (CFT)"

Free Boson CFT

Also known as "Tomonaga-Luttinger Liquid"
in condensed matter physics!

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 \quad \longleftrightarrow \quad \mathcal{L} = \frac{K}{2\pi}(\partial_\mu \theta)^2$$

"T duality"

$$\phi \sim \phi + \pi \qquad\qquad \theta \sim \theta + 2\pi$$

K: "Luttinger parameter"
corresponds to the
"compactification radius"
in hep-th literature

$$\tilde{\phi} \equiv \frac{\phi}{\sqrt{K}} \qquad \tilde{\theta} \equiv \sqrt{K}\theta$$

$$\tilde{\phi} \sim \tilde{\phi} + \frac{\pi}{\sqrt{K}} \qquad \tilde{\theta} \sim 2\pi\sqrt{K}$$

This Talk

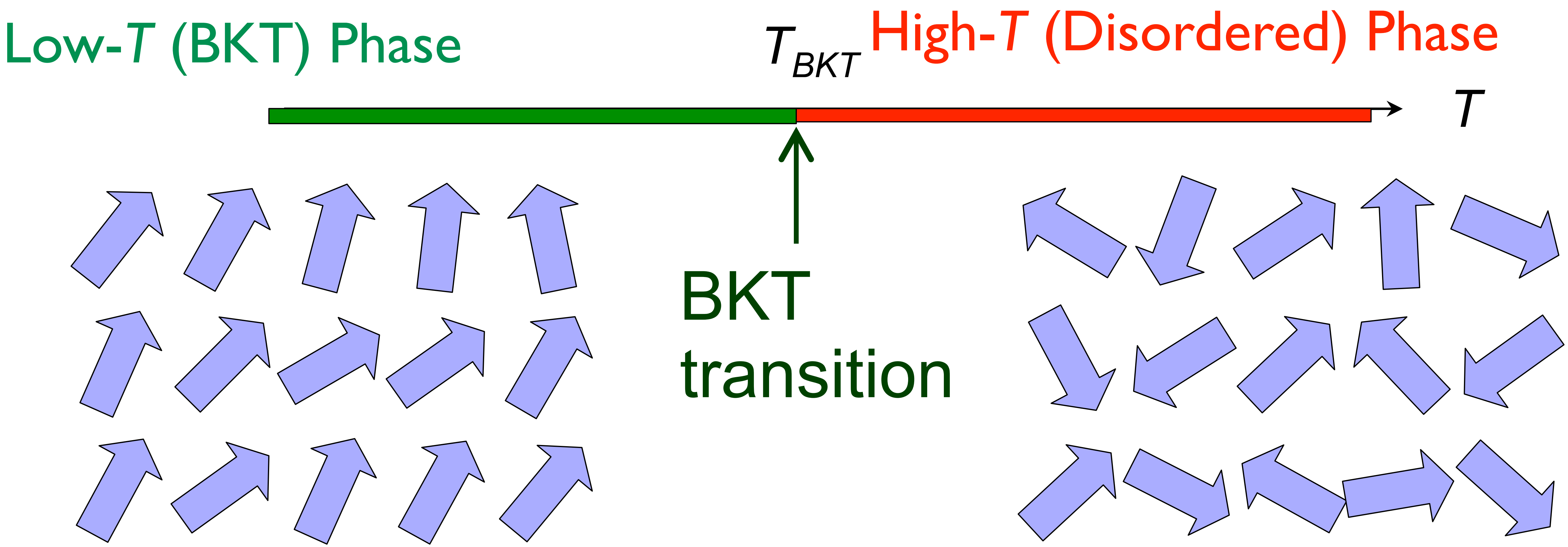
I will discuss some of the applications of the CFT
to condensed matter/statistical physics (in 1+1 Dim)
focusing on
free boson CFT (aka Tomonaga-Luttinger Liquid)

Most of the "theory" part is rather old

But we are seeing interesting developments,
thanks to recent advances in numerical algorithms

BKT Transition

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$ Classical 2D XY model



$$\vec{s}_j = (\cos\theta_j, \sin\theta_j)$$

$$\langle \vec{s}_i \cdot \vec{s}_j \rangle \propto \left(\frac{1}{r} \right)^{\eta}$$

power-law decay

$$\langle \vec{s}_i \cdot \vec{s}_j \rangle \propto \exp\left(-\frac{r}{\xi} \right)$$

exponential decay

Scientific Background on the Nobel Prize in Physics 2016

TOPOLOGICAL PHASE TRANSITIONS AND TOPOLOGICAL PHASES OF MATTER

compiled by the Class for Physics of the Royal Swedish Academy of Sciences

Prototype: Berezinskii-Kosterlitz-Thouless Transition

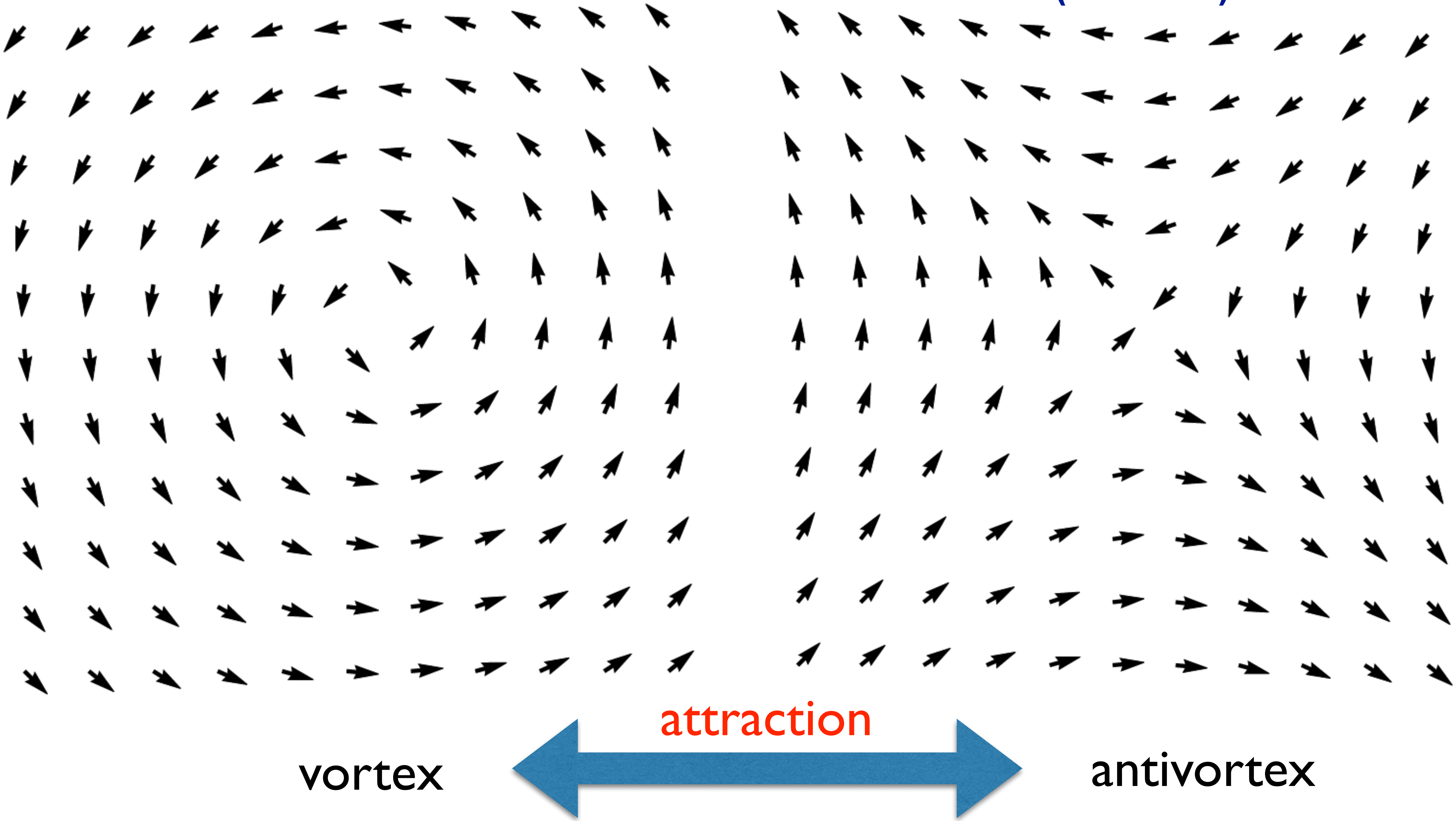
Canonical model: 2D classical XY model

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Vortex in the 2D XY model

XY spin goes back to itself by 2π-rotation ⇒

existence of defect (vortex)



BKT Transition

Low-*T* phase：vortex and antivortex form a pair

cf.) formation of atoms by nuclei and electrons

vortices are effectively absent at lengthscales larger than the pair size

$$\langle \vec{s}_i \cdot \vec{s}_j \rangle \propto \left(\frac{1}{r} \right)^{\eta}$$

High-*T* phase：vortices/antivortices dissociate from pairs and move freely

cf.) plasma state formed by dissociation of nuclei/electrons

$$\langle \vec{s}_i \cdot \vec{s}_j \rangle \propto \exp\left(-\frac{r}{\xi} \right)$$

Sine-Gordon Field Theory for BKT

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 - y_K (\partial_\mu \phi)^2 + y_V V$$

θ angle of the XY spin ⟷ ϕ

dual (mutually non-local)

$$V = \cos 2\phi = \frac{1}{2}\left(e^{2i\phi} + e^{-2i\phi}\right)$$

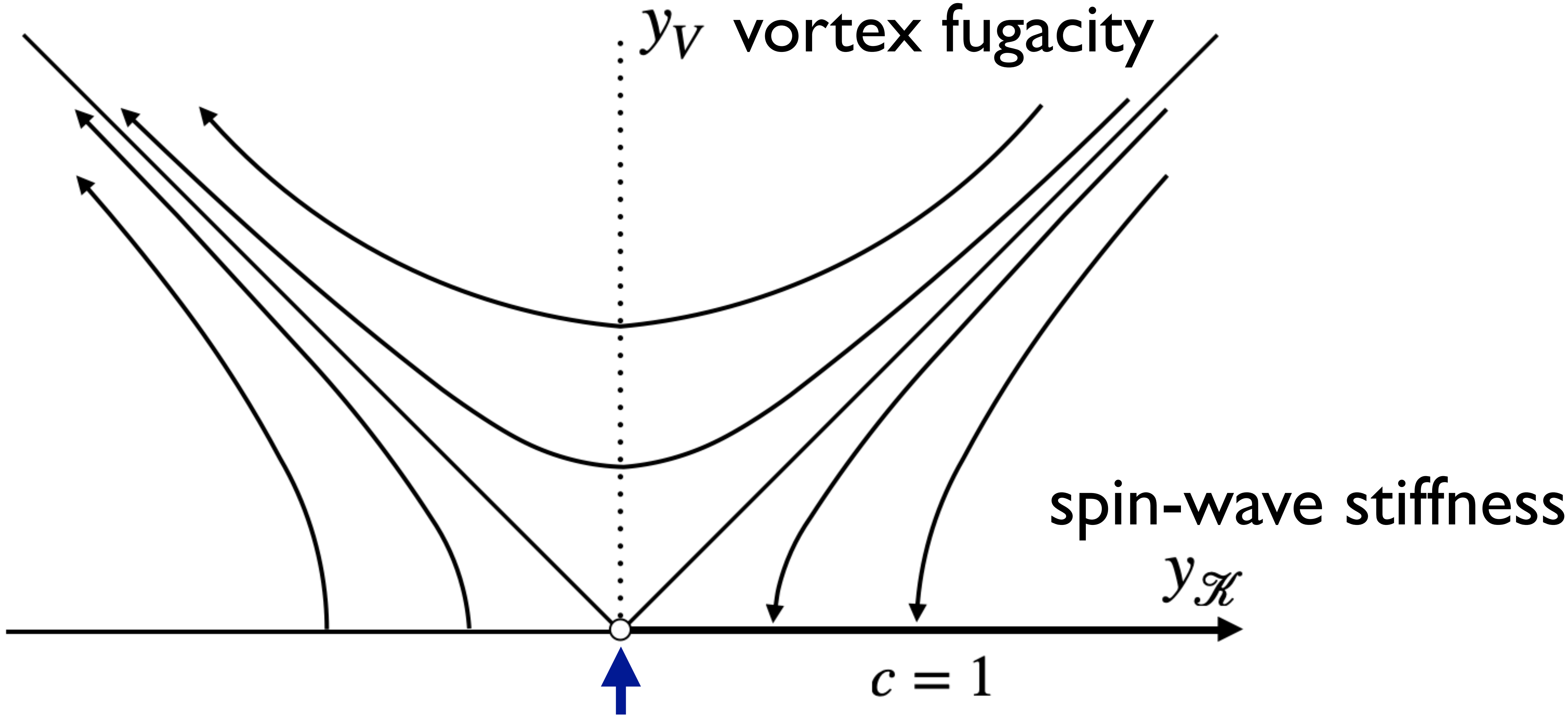
scaling dimension 2/*K*
marginal at *K*=2

Single vortex creation / annihilation operator

y_V vortex fugacity

y_K renormalization of Luttinger parameter *K*

Kosterlitz RG Flow



Luttinger parameter *K*=2 (vortex operator marginal)

Why CFT?

Free boson CFT is exactly solvable (Gaussian free field)

Actually, the BKT transition was predicted and the Kosterlitz RG flow was derived without CFT techniques

Nevertheless, numerical study of the BKT transition is rather challenging (even though it is “well-understood”) and CFT ideas turn out to be practically very useful!

2D XY Model

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

- Classical spin model with positive Boltzmann weight
  $\Rightarrow$ no sign problem

- Just 2 dimensions
- Efficient cluster algorithms available

Easily studied with Monte Carlo, right?

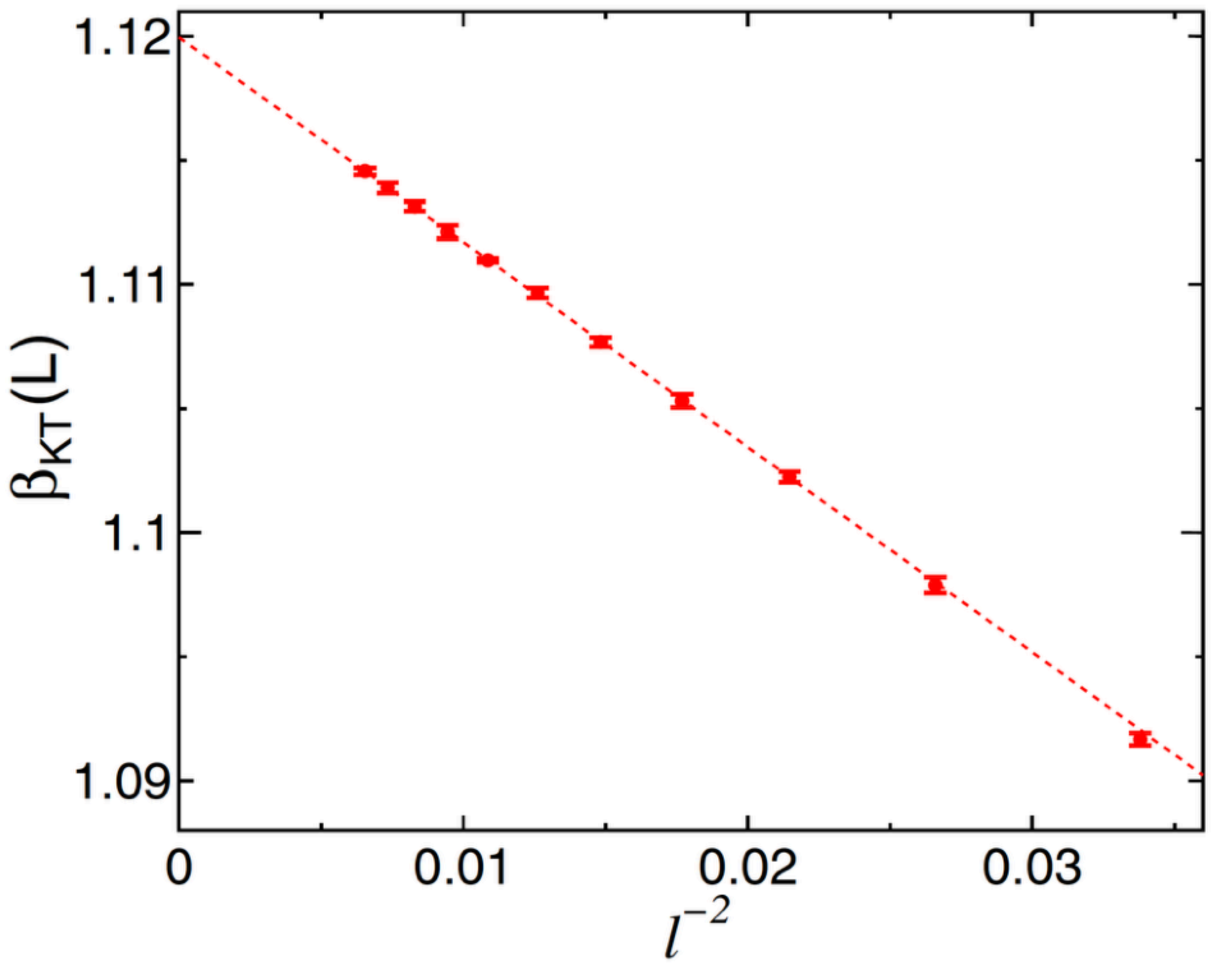
Journal of the Physical Society of Japan **81** (2012) 113001 LETTERS
http://dx.doi.org/10.1143/JPSJ.81.113001

Large-Scale Monte Carlo Simulation of Two-Dimensional Classical XY Model Using Multiple GPUs

Yukihiro KOMURA* and Yutaka OKABE†

Department of Physics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan

(Received August 27, 2012; accepted September 24, 2012; published online October 12, 2012)



finite-size scaling of T_c

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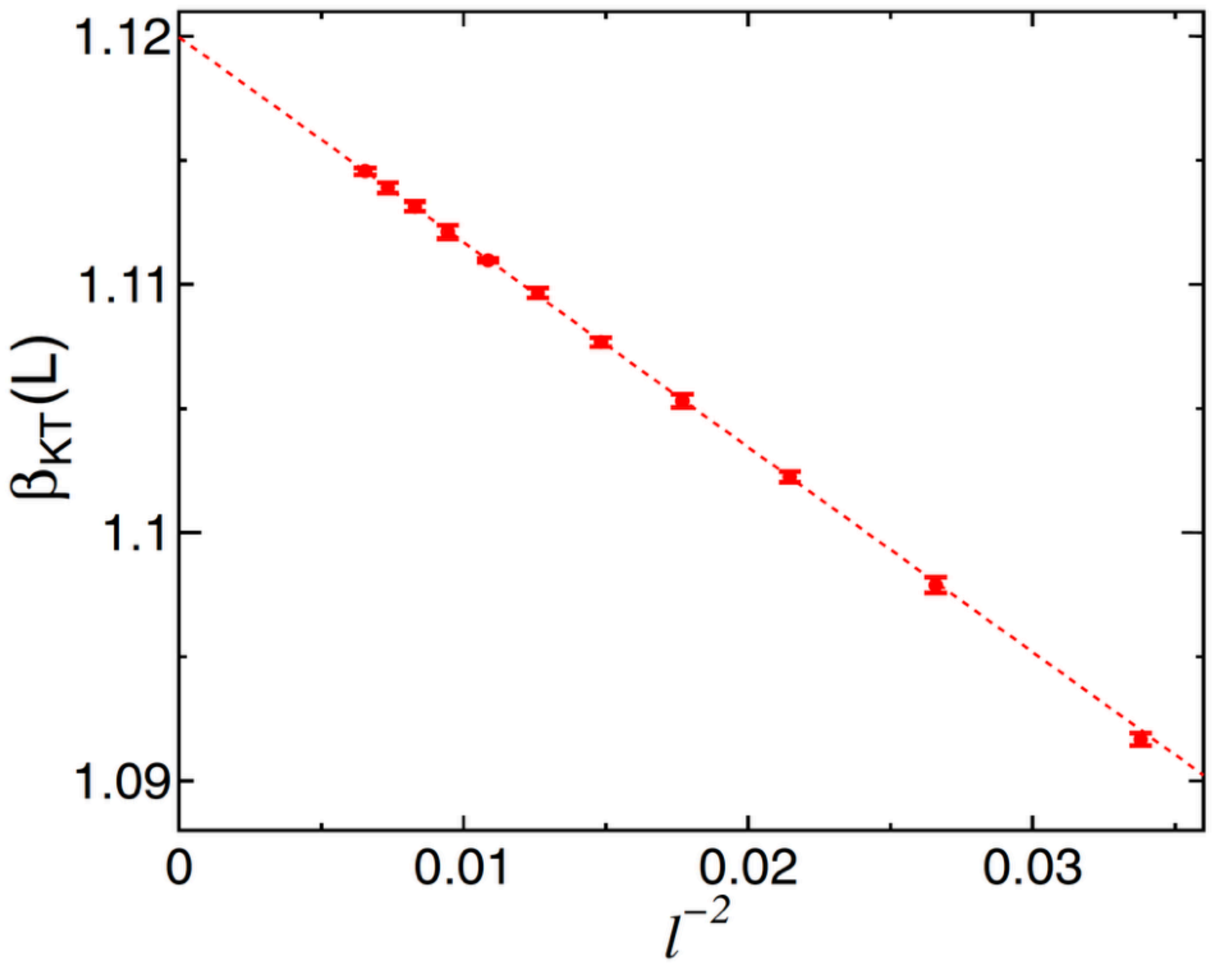

finite-size scaling of T_c

$$l = \ln bL$$

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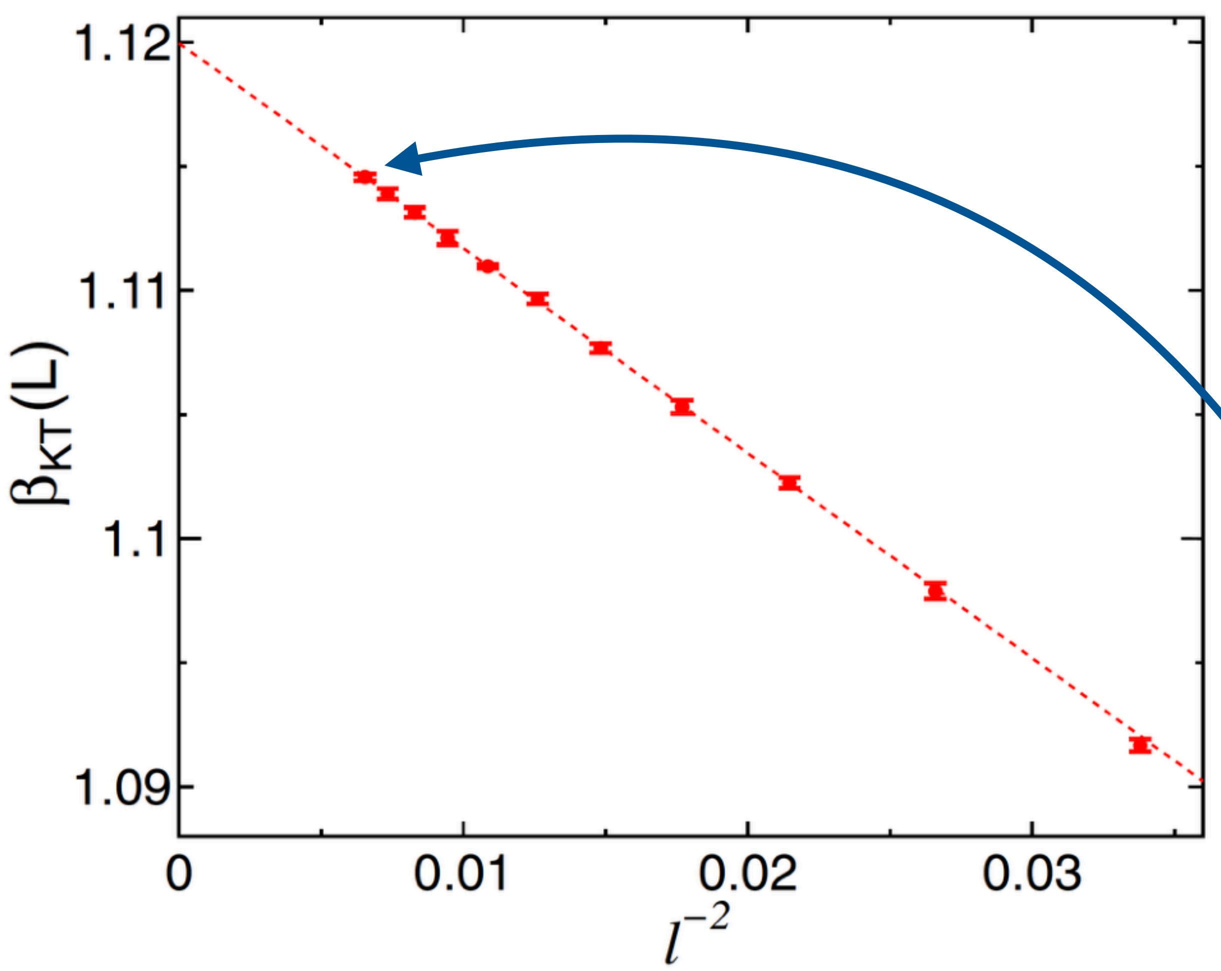
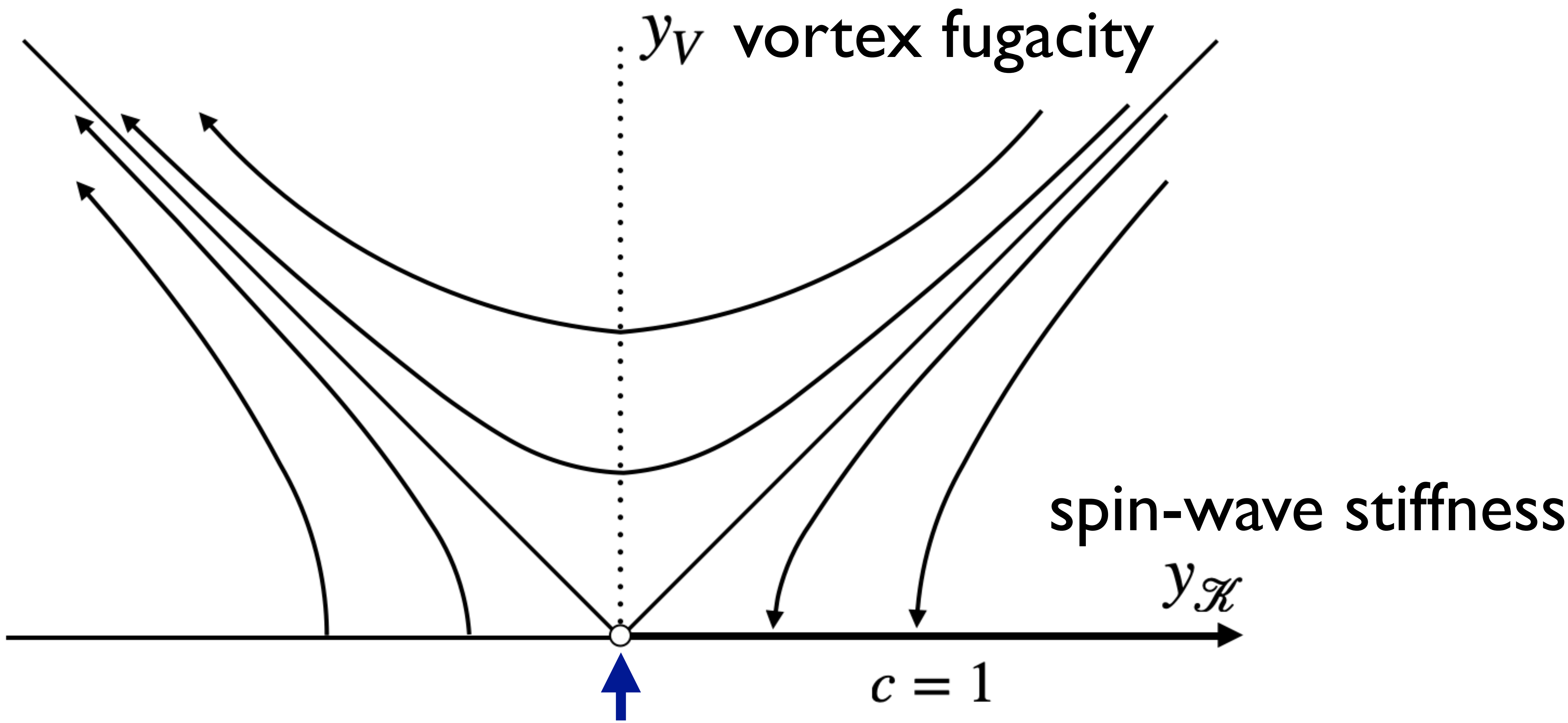


finite-size scaling of T_c

$$l = \ln bL$$

largest system:
L=65536
calculation using 256GPUs

Kosterlitz RG Flow



Luttinger parameter *K*=2 (vortex operator marginal)

BKT transition: $y_V = y_{\mathcal{K}} = g$

$$\frac{dg}{dl} = -g^2 \qquad g \sim \frac{1}{l} \sim \frac{1}{\ln L}$$

slow decay

↓

log-corrections

Should We Care?

We already know the exact exponents at the BKT transition, thanks to RG (or CFT)

But for "practical" purposes, determination of the critical point (which is non-universal) can be important

The "brute-force" calculation already gives T_c for 3 digits, which might be enough in practice

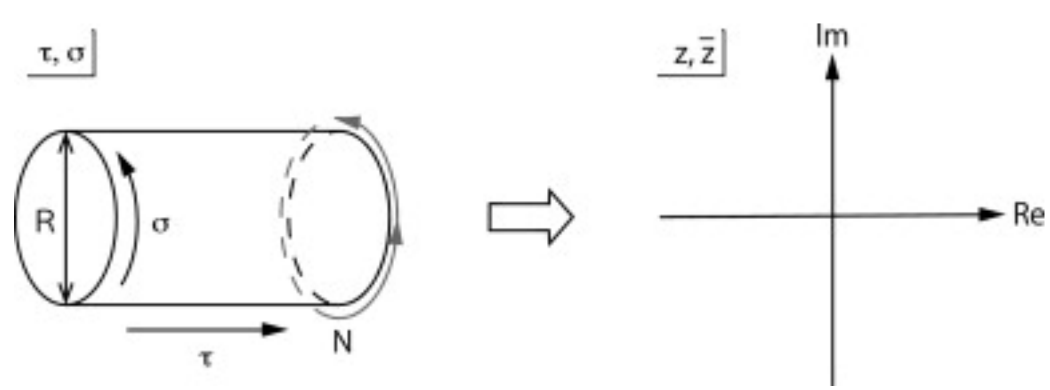
However, in order to elucidate more complicated systems, it is highly desirable to develop a more efficient method (even the universality class can be controversial!)

CFT for 1+1D Quantum Systems

Interestingly, the difficulty of the logarithmic corrections in numerics was "solved" first for 1+1D quantum systems

In condensed matter physics, a quantum system are usually defined in terms of a Hamiltonian

In CFT, we can formulate the spectrum of the Hamiltonian by a conformal mapping between a cylinder and infinite plane



State-Operator Correspondence

The conformal mapping also implies a one-to-one correspondence between quantum states and operators

Energy eigenvalue of a finite system (under periodic b.c.)

$$E_n - E_0 = \frac{2\pi}{L} x_n$$

[Cardy 1984]

L: system size (length)
x_n: scaling dimension of the (primary) operator

Scaling dimensions (critical exponents) can be extracted from the spectrum!

Spectrum of Perturbed CFT

$$\mathcal{H} = \mathcal{H}_{\mathrm{CFT}} + \sum_n y_n \int \Phi_n(\mathbb{r})\, d\mathbb{r}$$

[Cardy 1986]

$$E_n - E_0 = \frac{2\pi}{L}\left(x_n + \sum_m c_{nnm} y_m L^{2-x_m} + \ldots \right)$$

c_{lmn} : Operator Product Expansion coefficient

$$\Phi_l \Phi_m \sim \sum_n c_{lmn} \Phi_n$$

The coefficients of the perturbations can be also extracted from the finite-size spectrum!

BKT Transition in S=1/2 XXZ Chain

$$\mathcal{H} = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

Single vortex creation/annihilation operator

$$\cos 2\phi \sim (-1)^j \vec{S}_j \cdot \vec{S}_{j+1}$$

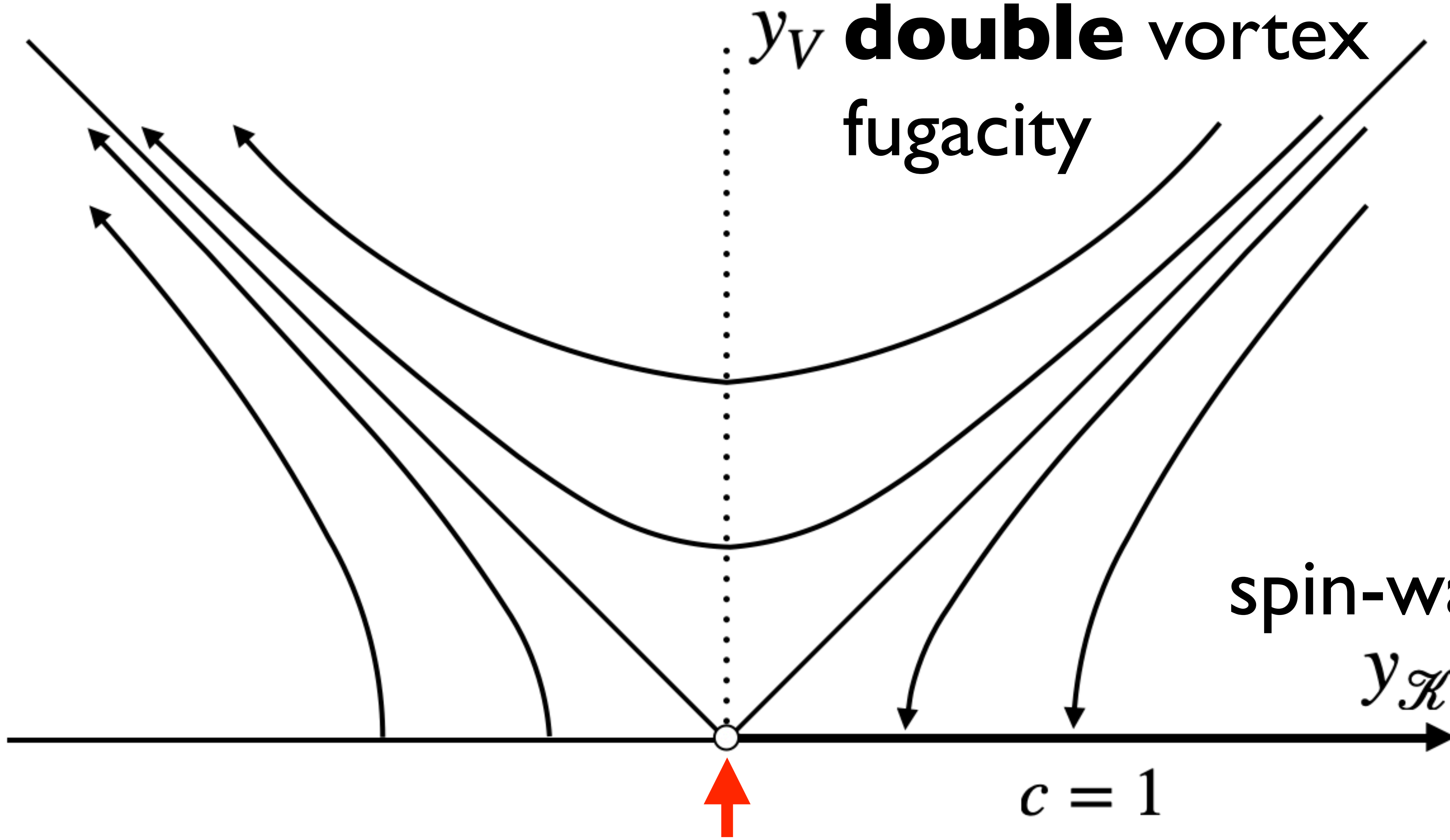
$$\sin 2\phi \sim (-1)^j S_j^z$$

Forbidden in Hamiltonian by the translation symmetry!
(Haldane 1980 → "Haldane conjecture"
related to Lieb-Schultz-Mattis theorem etc.)

The leading (most relevant) perturbation is thus
double vortex creation/annihilation op. $\cos 4\phi$

BTK Transition in S=1/2 XXZ Chain

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 - y_{\mathcal{K}}(\partial_\mu \phi)^2 + y_V V \qquad V = \cos 4\phi$$



Luttinger parameter ***K*=1/2**
(**double** vortex operator marginal)

BKT Transition in S=1/2 XXZ Chain

$$\mathcal{H} = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

BKT transition at Δ=1 (SU(2) symmetric point)

IR fixed point at the BKT transition:

Free boson (Tomonaga-Luttinger Liquid) at *K*=1/2

equivalent to Level 1 SU(2) Wess-Zumino-Witten

Effective theory in the vicinity of the BKT transition

$$\mathcal{L} = \mathcal{L}_{k=1}^{\mathrm{WZW}} + g \mathbb{J}^L \cdot \mathbb{J}^R + t \left(-\frac{1}{2} J_+^L J_+^R - \frac{1}{2} J_-^L J_+^R + J_z^L J_z^R \right)$$

$$y_V = g + t, y_{\mathcal{K}} = g - t$$

BKT transition ⇔ *t*=0 ⇔ SU(2) symmetry

Level Spectroscopy

Determination of the critical point from the finite-size spectrum [Okamoto-Nomura 1994, …]

"Double vortex" BKT transition at K=1/2 can be identified by SU(2) symmetry of the finite-size spectrum!

State-operator correspondence and Finite-Size Scaling in CFT [Cardy 1984, 1986]

$$E_n - E_0 = \frac{2\pi}{L}\left(x_n + \sum_m c_{nnm} y_m L^{2-x_m} + \dots \right)$$

BKT transition ⇔

Energy levels form SU(2) singlet, triplet, …

1D S=1/2 XXZ vs 2D Classical XY

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 - y_K (\partial_\mu \phi)^2 + y_V V$$

Nomura-Kitazawa 1998

S=1/2 XXZ

K=1/2 (SU(2)$_1$ WZW)

$V \sim \cos 4\phi$

double vortex op.

SU(2) triplet (degenerate at BKT)

$n^x \sim \cos\theta,\ n^y \sim \sin\theta,\ n^z \sim \sin 2\phi$

$2\phi \to \phi$

S=1 XXZ

Classical XY

K=2

$V \sim \cos 2\phi$

single vortex op.

hidden SU(2) triplet

$\cos 2\theta, \sin 2\theta, \sin\phi$

half-vortex op.
(eigenstate under antiperiodic b.c.)

J. Phys. A: Math. Gen. **31** (1998) 7341–7362. Printed in the UK PII: S0305-4470(98)88524-7

$SU(2)/Z_2$ symmetry of the BKT transition and twisted boundary condition

Kiyohide Nomura† and Atsuhiro Kitazawa†‡

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‡ Department of Physics, Tokyo Institute of Technology, Tokyo 152, Japan

Received 20 October 1997

Abstract. The Berezinskii–Kosterlitz–Thouless (BKT) transition, the transition of the two-dimensional sine-Gordon model, plays an important role in low-dimensional physics. We relate the operator content of the BKT transition to that of the $SU(2)$ Wess–Zumino–Witten model, using twisted boundary conditions. With this method, in order $k-1$ to determine the BKT critical point, we can use the level crossing of the lower excitations instead of those for the periodic boundary case, thus the convergence to the transition point is highly improved. We verify the efficiency of this method by applying it to the $S=1, 2$ spin chains.

Level Spectroscopy for 2D Stat Mech

Level spectroscopy has been developed for quantum 1D,
but why not for 2D stat mech models (such as XY)??

1D quantum Hamiltonian ⇔

Transfer matrix for 2D stat mech

Continuous spin:

"infinite dimensional local Hilbert space"

can be discretized, but still the dimension is too large for exact diagonalization arXiv.org > cond-mat > arXiv:2105.11460

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Condensed Matter > Statistical Mechanics

[Submitted on 24 May 2021]

Resolving the Berezinskii-Kosterlitz-Thouless transition in the 2D XY model with tensor-network based level spectroscopy

Atsushi Ueda, Masaki Oshikawa

Berezinskii-Kosterlitz-Thouless transition of the classical XY model is re-investigated, combining the Tensor Network Renormalization (TNR) and the Level Spectroscopy method based on the finite-size scaling of the Conformal Field Theory. By systematically analyzing the spectrum of the transfer matrix of the systems of various moderate sizes which can be accurately handled with a finite bond dimension, we determine the critical point removing the logarithmic corrections. This improves the accuracy by an order of magnitude over previous studies including those utilizing TNR. Our analysis also gives a visualization of the celebrated Kosterlitz Renormalization Group flow based on the numerical data.

Subjects: **Statistical Mechanics (cond-mat.stat-mech)**
Cite as: arXiv:2105.11460 **[cond-mat.stat-mech]**
(or arXiv:2105.11460v1 **[cond-mat.stat-mech]** for this version)

Submission history

From: Atsushi Ueda [view email]
[v1] Mon, 24 May 2021 18:00:01 UTC (976 KB)

Atsushi Ueda

Level Spectroscopy for 2D Stat Mech

Level spectroscopy has been developed for quantum 1D, but why not for 2D stat mech models (such as XY)??

1D quantum Hamiltonian ⇔

Transfer matrix for 2D stat mech

Continuous spin: series expansion of Boltzmann weight

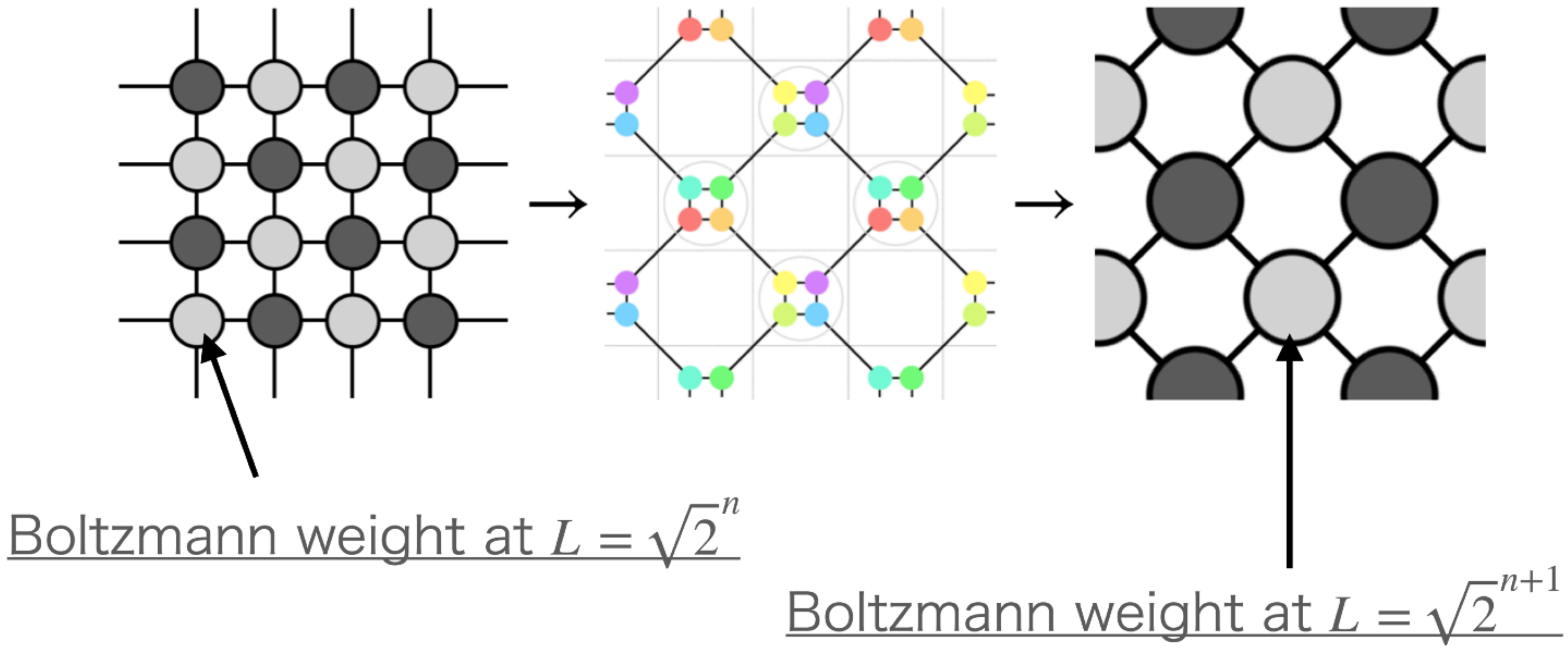
$$e^{\beta \cos(\theta_i - \theta_j)} = \sum_{n=-\infty}^{\infty} e^{in(\theta_i - \theta_j)} I_n(\beta),$$

the series may be truncated to $-15 \leqq n \leqq 15$

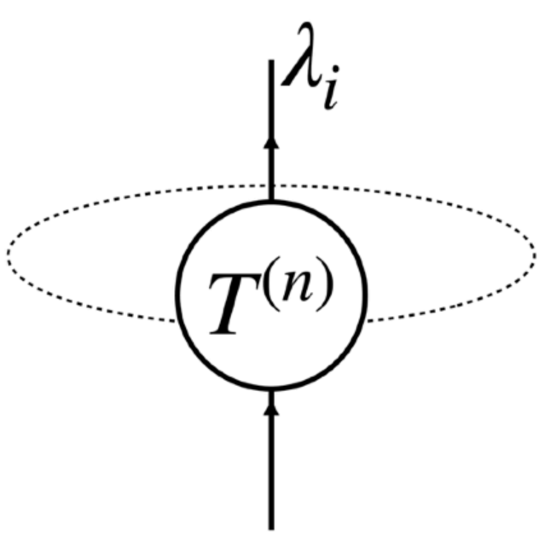
Transfer matrix still "too large" to be diagonalized
⇒ we utilize Tensor Network Renormalization

(We used Loop-TNR)

TNR Construction of Transfer Matrix



after n steps, a single tensor represents a square block of linear size $L = \sqrt{2}^n$



contract horizontal indices

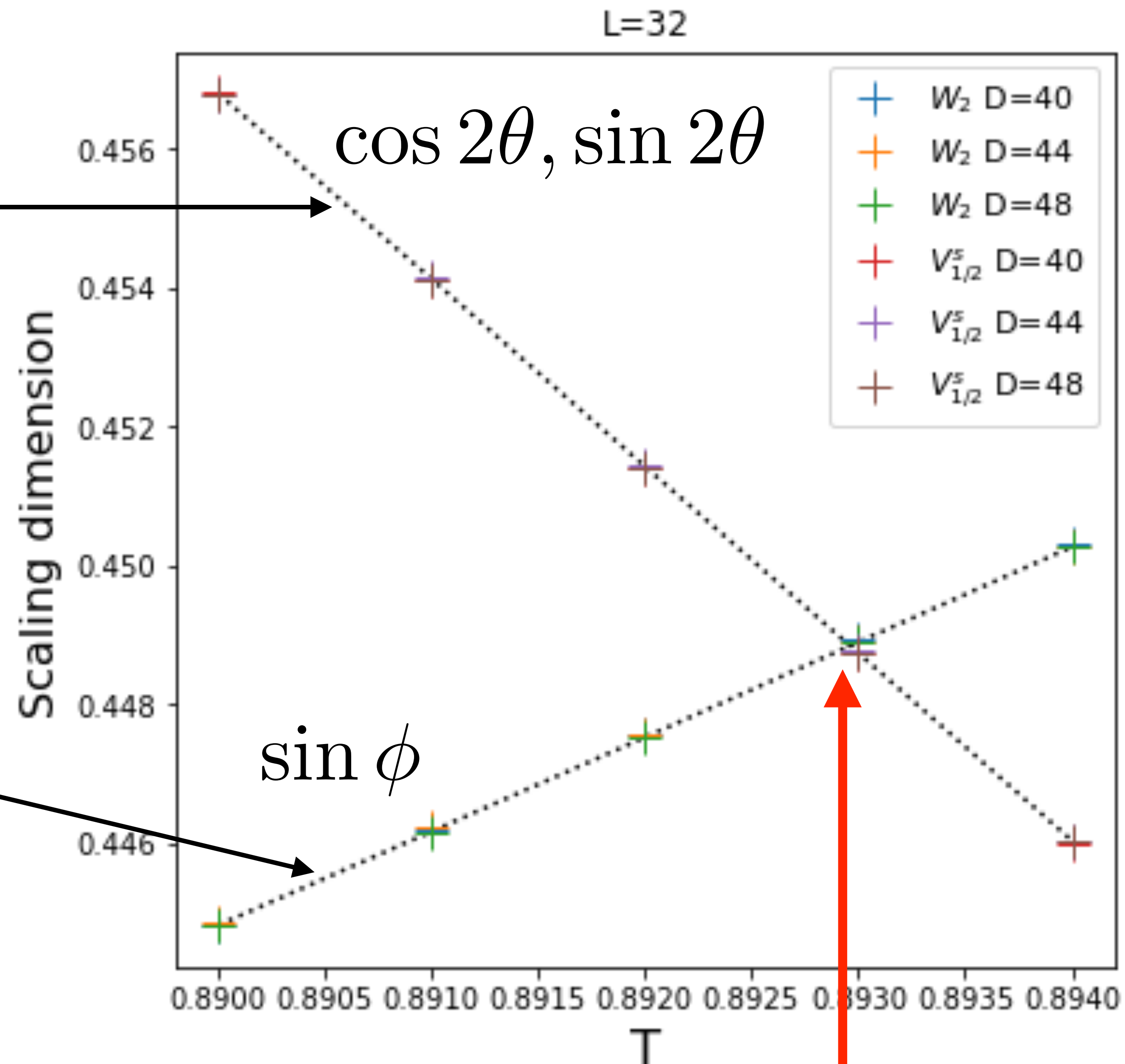
⇒ transfer matrix in vertical direction

$$\lambda_i = e^{-LE_i(L)}$$

Identifying T_c with Level Crossing

"spin-wave"
excited states
under periodic b.c.

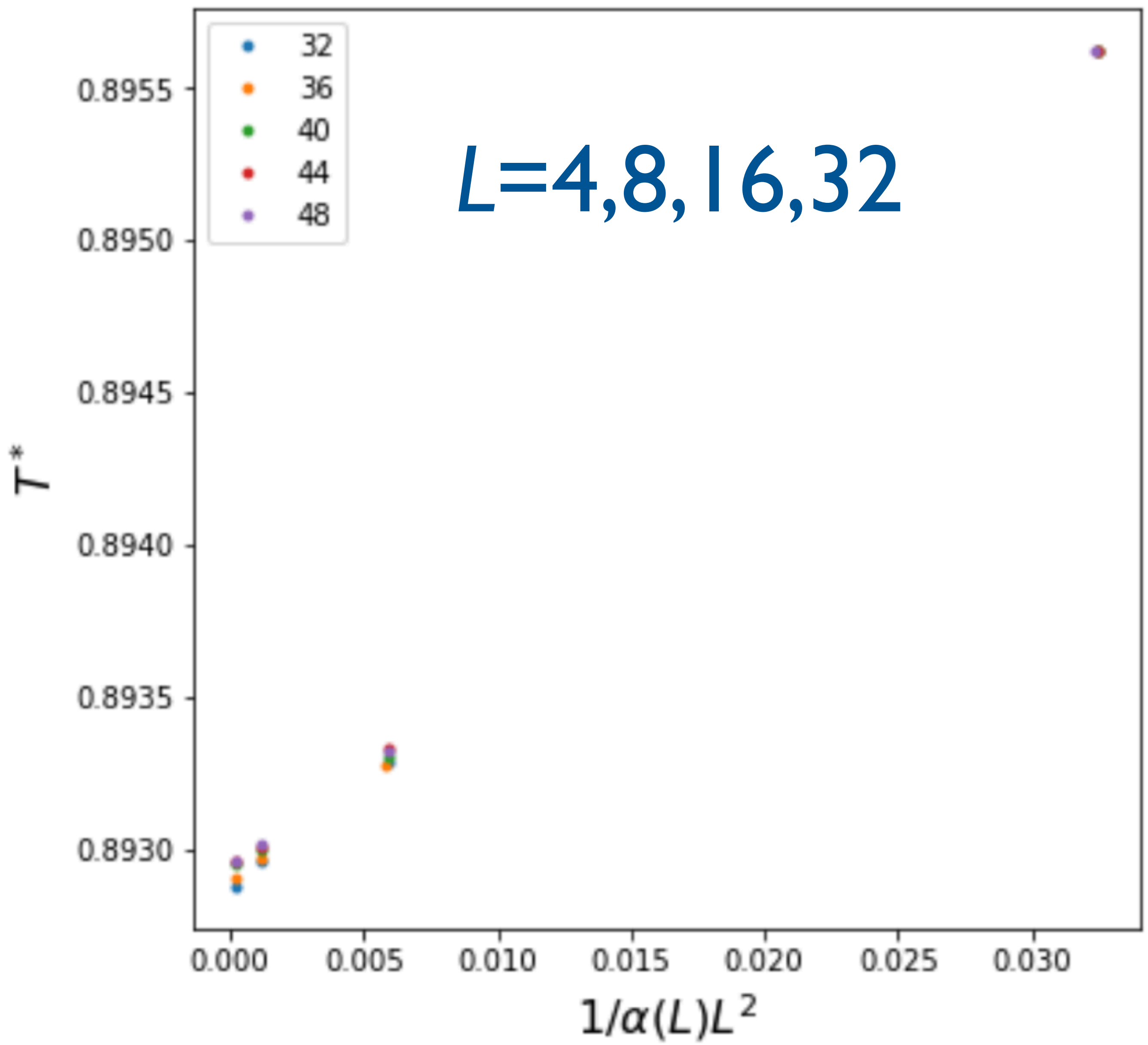
ground state under
antiperiodic b.c.



This procedure eliminates
logarithmic corrections
to all orders in g

extra degeneracy
forming SU(2) triplet
$\sim$ BKT transition

Remaining Finite-Size Effect



Level crossing point weakly depends on the system size *L*

Effect of irrelevant perturbations

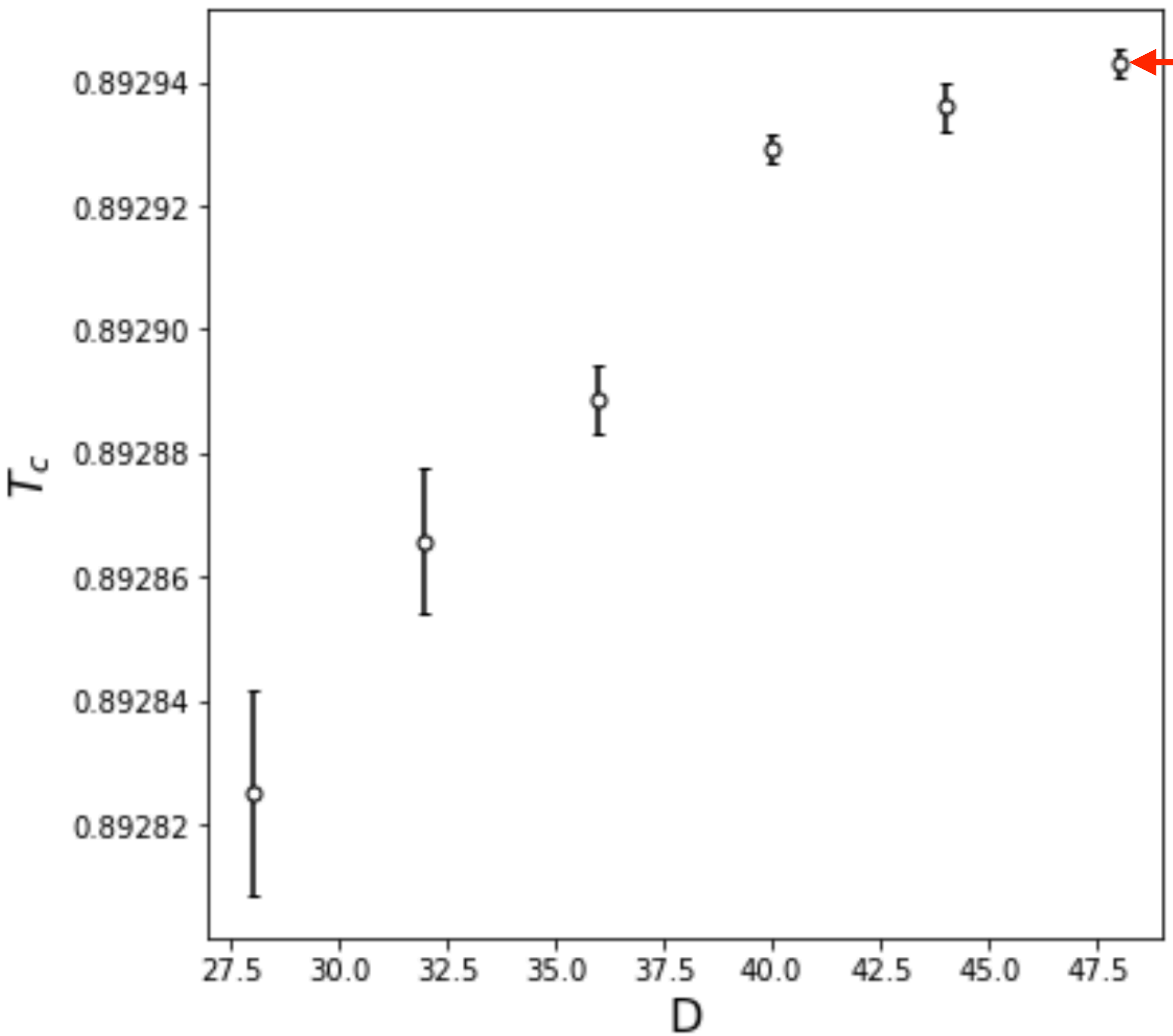
$$T^2, \bar{T}^2, T\bar{T}, \ldots$$

T: holomorphic part of the energy-momentum tensor

$$T^* \sim T_c + \text{const.}\frac{1}{L^2}$$

Extrapolate to *L*=∞

Dependence on Bond Dimension *D*



Our final estimate
T_c=0.892943(2)

Effect of Finite Bond-Dimension

Finite bond dimension *D* ⇔ finite "correlation length"

$$\xi_D \sim 0.3 D^{\kappa}$$

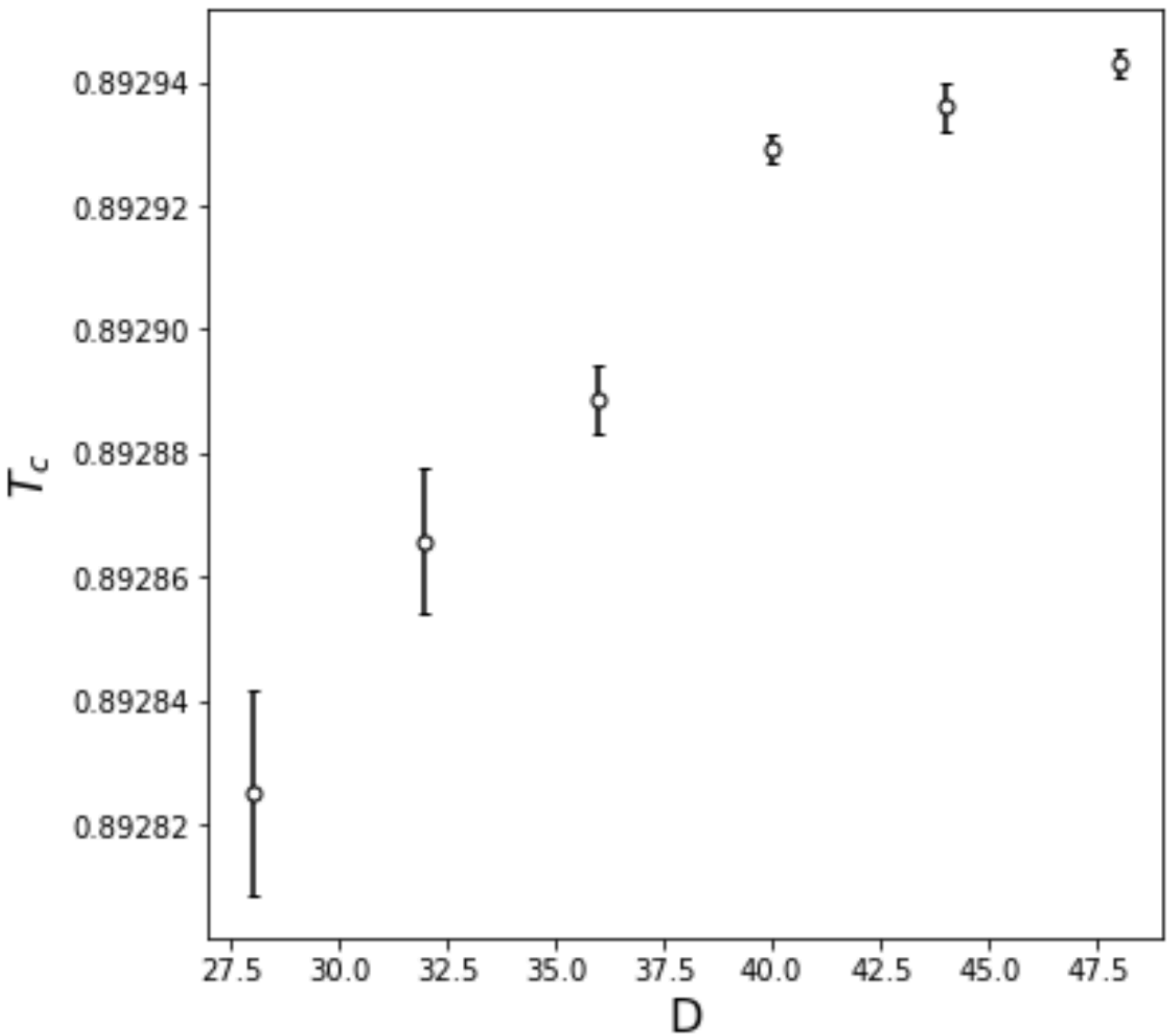
$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}} + 1\right)}$$

[Pollmann et al. 2008]

$\xi_D > L$ low-energy finite-size spectrum almost exact!

$\xi_D < L$ low-energy spectrum still reasonably accurate, but some error due to the finite *D*

Tc dependence on *D*



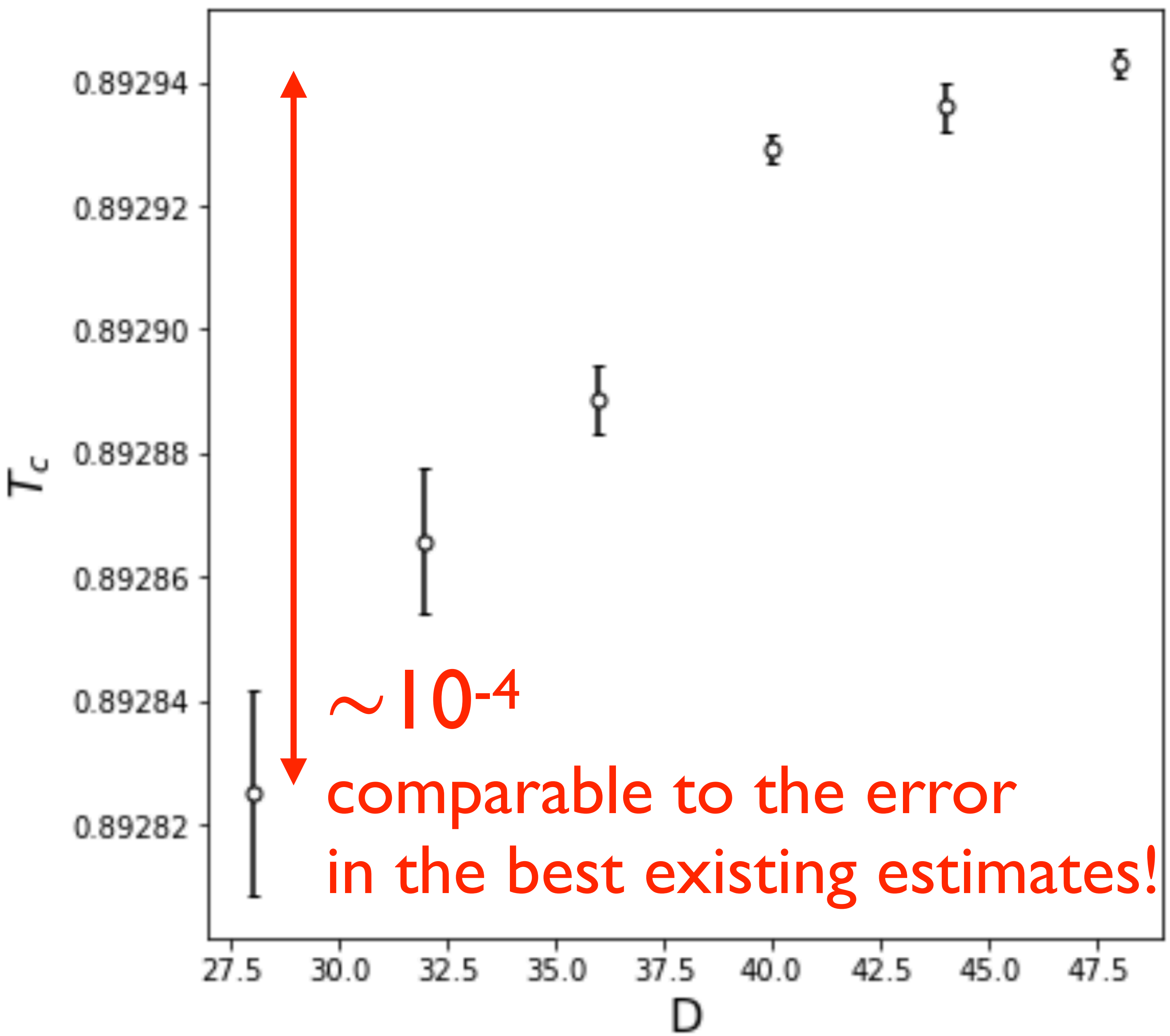
D=48 gives ξ～54

enough for up to *L*=32

D=28 gives ξ～26

too small for *L*=32
BUT….

Tc dependence on *D*



D=48 gives ξ∼54

enough for up to *L*=32

D=28 gives ξ∼26

too small for *L*=32

BUT….

Error in (Loop-)TNR

(Loop-)TNR is often used to construct the "fixed point" tensor, which would describe the large scale behaviors, by iterating TNR many times

This approach has given accurate results for large systems, but small errors due to the finite bond dimension remain

[A. Ueda & M.O., in preparation]

In our approach, we study the spectrum of finite-size systems with TNR. TNR is almost exact when the system size is less than the effective correlation length.

TNR calculation of the finite-size spectrum
+ Level Spectroscopy → better accuracy

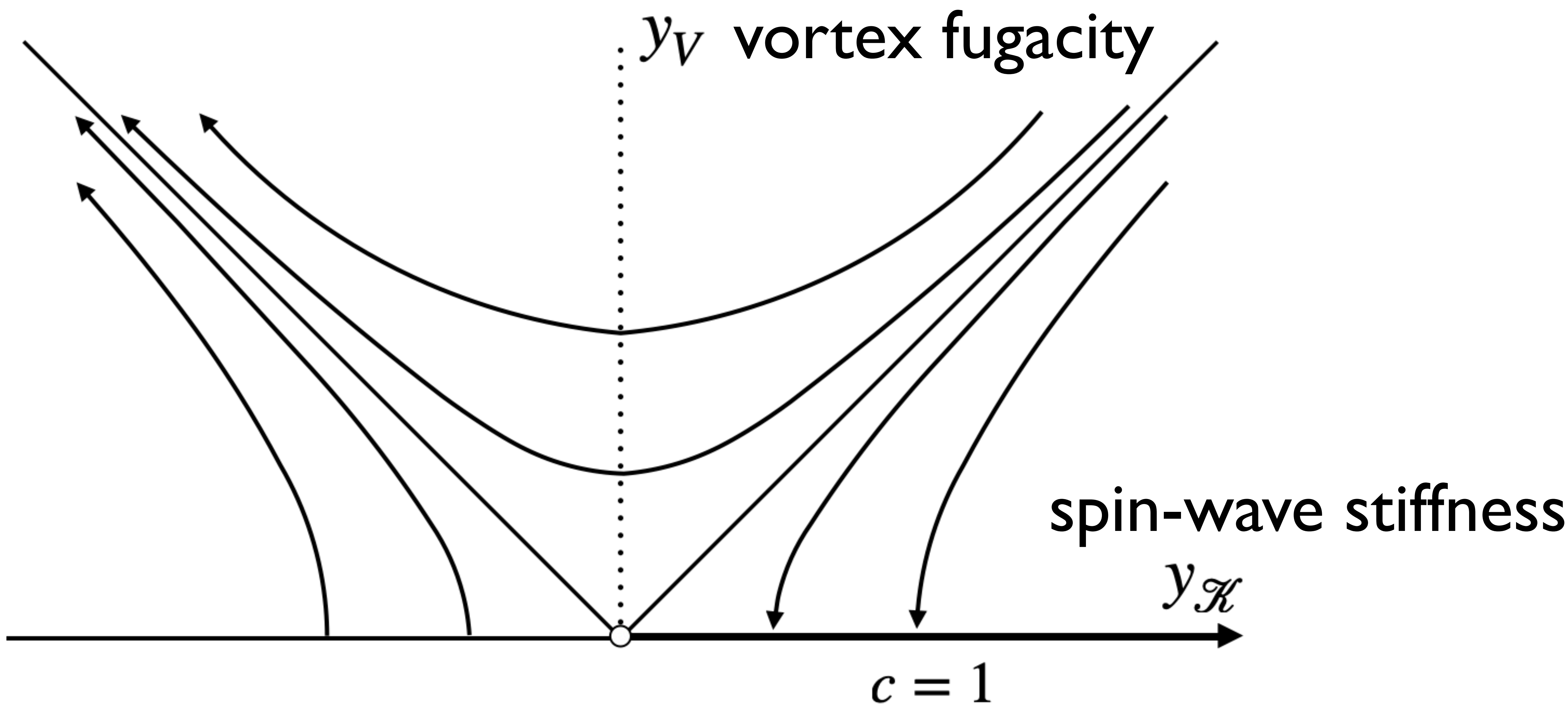
Estimates of T_c

Monte Carlo(1979)[35]	0.89
Monte Carlo(2005)[36]	0.8929
Monte Carlo(2012)[37]	0.89289
Monte Carlo(2013)[38]	0.8935
Series expansion(2009)[39]	0.89286
HOTRG(2014)[40]	0.8921
VUMPS(2019)[41]	0.8930
HOTRG(2020)[42]	0.89290(5)
present work	0.892943(2)

TABLE I. Comparison of the estimated critical temperature of the 2D classical XY model.

Kosterlitz RG Flow

You must have seen this diagram many times….



Kosterlitz RG Flow

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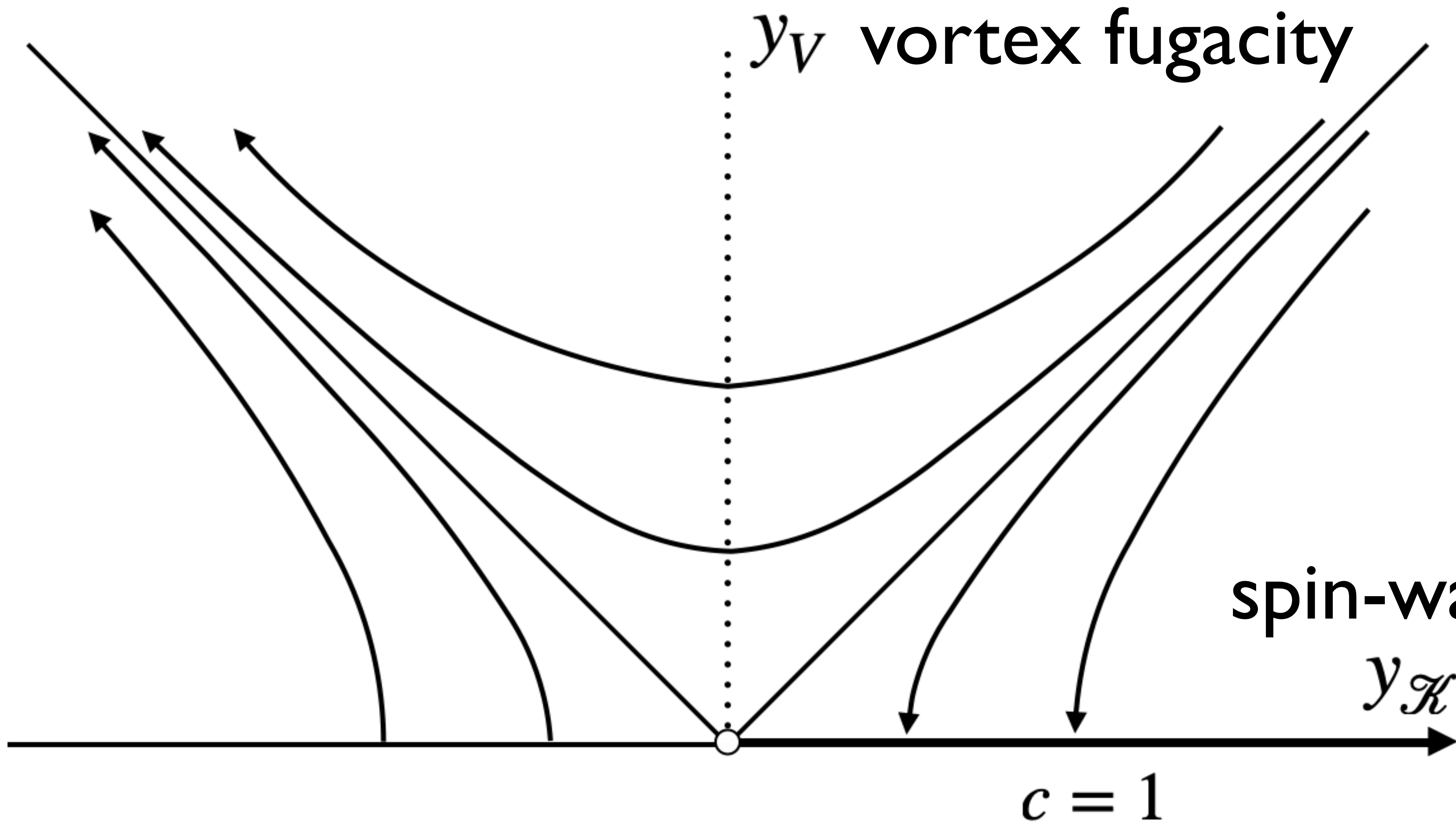

*but have you **really** seen the RG flow?*

ELSEVIER

Nuclear Physics B 522 [FS] (1998) 533–549

NUCLEAR PHYSICS B

Low energy effective Hamiltonian for the XXZ spin chain

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L.D. Landau Institute for Theoretical Physics, Kosygina 2, Moscow, Russia

Received 19 February 1998; accepted 18 March 1998

“vacuum energy” under the twisted boundary condition with the twist angle θ [notation clash…]

$$\delta^{\mathrm{RG}} = -\frac{1}{12}\left\{1+\frac{3}{8}g_{\|}g_{\perp}^{2}\right\} + \frac{s^2}{2}\left\{1-\frac{g_{\|}}{2}+\frac{1}{4}g_{\perp}^{2}-\frac{7}{32}g_{\|}g_{\perp}^{2}\right\}$$
$$+\frac{|s|}{16}\left\{2g_{\perp}^{2}-g_{\|}g_{\perp}^{2}\right\} + \frac{\theta^2}{2}\left\{1+\frac{g_{\|}}{2}+\frac{g_{\|}^{2}}{4}-\frac{g_{\perp}^{2}}{4}+\frac{g_{\|}^{3}}{8}-\frac{g_{\|}g_{\perp}^{2}}{32}\right\} + O(g^4), \qquad (4.6)$$

Extrapolating Lukyanov's Result

In our notations, the energy levels would be given by

$$E_n - E_0 = \frac{2\pi}{L} x_n$$

$e^{\pm i2\theta}$ $$x_{2,0} = \frac{1}{2} - \frac{y_{\mathcal{K}}}{4} + (\frac{1}{4} - \frac{11}{64} y_{\mathcal{K}}) y_V^2,$$

$e^{\pm i\phi}$ $$x_{0,1/2} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} + \frac{1}{8}(y_{\mathcal{K}}^2 - y_V^2) + \frac{y_{\mathcal{K}}^3}{16} - \frac{1}{64} y_{\mathcal{K}} y_V^2.$$

They form two doublets, and no triplet is formed even on the BKT transition line $y_{\mathcal{K}} = y_V$?!

The two states corresponding to $e^{\pm i\phi}$ are mixed by the vortex perturbation $\cos 2\phi$ and split into two levels corresponding to $V^s_{1/2} = \sin\phi, V^c_{1/2} = \cos\phi$

Energy Levels up to 2nd Order

- split between $x_{V^s_{1/2}}, x_{V^c_{1/2}}$ should be odd in y_V
- SU(2) triplet should be formed on
  the BKT transition line $y_{\mathcal{K}} = y_V$

⇒ uniquely determines the energy levels up to $O(y^2)$

$$x_{W_{\pm 2}} = \frac{1}{2} - \frac{y_{\mathcal{K}}}{4} + \frac{1}{4} y_V^2,$$

$$x_{V^s_{1/2}} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} - \frac{y_V}{2} + \frac{1}{8}(y_{\mathcal{K}}^2 + 2 y_{\mathcal{K}} y_V - y_V^2),$$

$$x_{V^c_{1/2}} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} + \frac{y_V}{2} + \frac{1}{8}(y_{\mathcal{K}}^2 - 2 y_{\mathcal{K}} y_V - y_V^2),$$

Obtaining Running Coupling Constants

$$x_{W_{\pm 2}} = \frac{1}{2} - \frac{y_{\mathcal{K}}}{4} + \frac{1}{4}y_V^2,$$

$$x_{V^s_{1/2}} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} - \frac{y_V}{2} + \frac{1}{8}(y_{\mathcal{K}}^2 + 2y_{\mathcal{K}}y_V - y_V^2),$$

$$x_{V^c_{1/2}} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} + \frac{y_V}{2} + \frac{1}{8}(y_{\mathcal{K}}^2 - 2y_{\mathcal{K}}y_V - y_V^2),$$

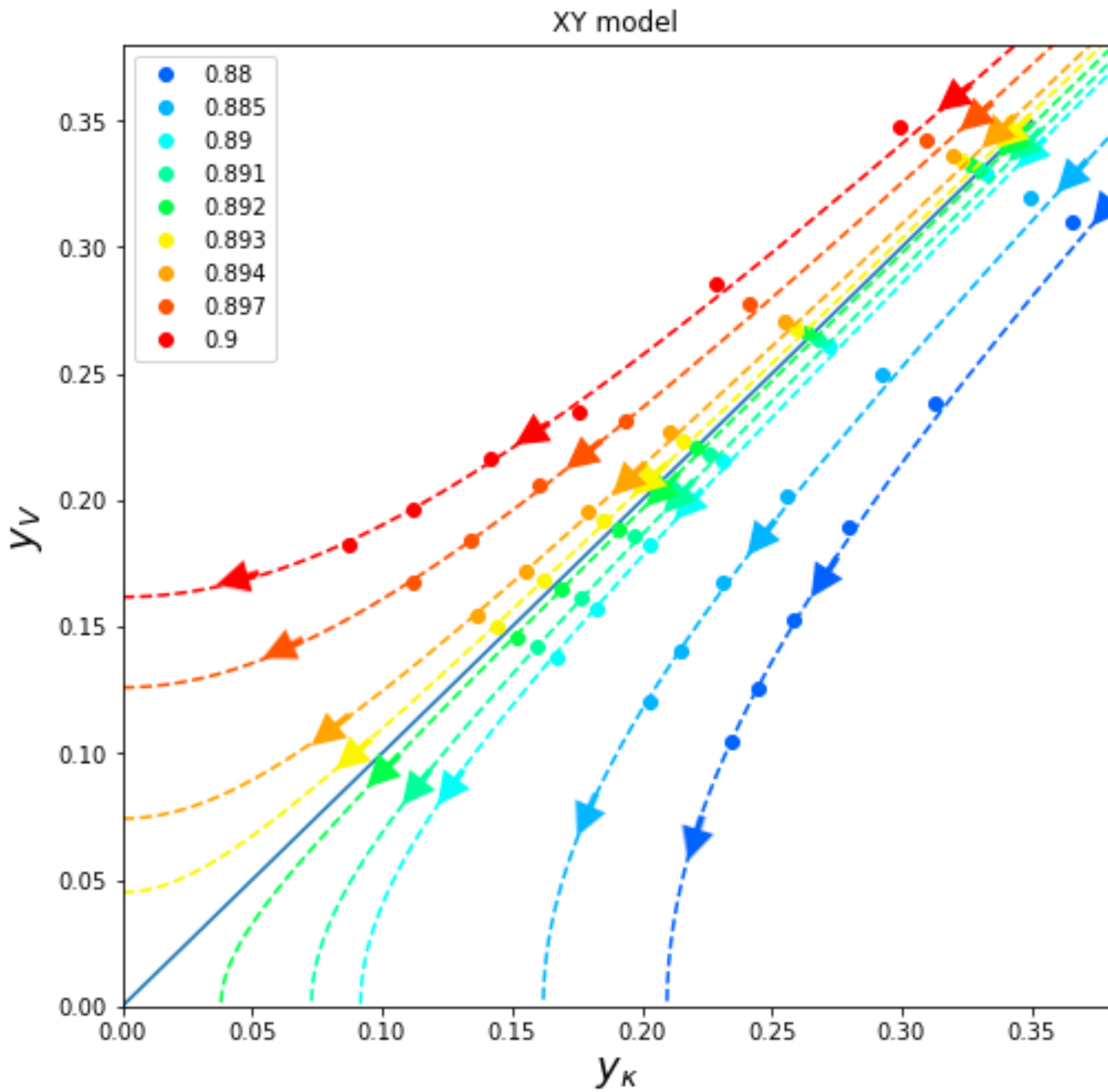
$$y_{\mathcal{K}} \sim 2 - 4x_{W_{\pm 2}} + (x_{V^c_{1/2}} - x_{V^s_{1/2}})^2,$$

$$y_V \sim (x_{V^c_{1/2}} - x_{V^s_{1/2}})/(1 - \frac{1}{2}y_{\mathcal{K}}),$$

We can estimate y_K & y_V from the finite-size energy levels

Less accuracy than T_c, but we can apply to larger systems (up to L=512)

Visualization of Kosterlitz RG Flow!



L=16
32
64
128
256
512

Conclusions (Part 1)

TNR + Level Spectroscopy (finite size scaling of CFT)
allows
- super accurate determination of BKT critical point
- visualization of Kosterlitz RG flow by extraction of running coupling constants from the spectrum

for continuous valued 2D classical spin system such as XY model

Future: extension/application to more nontrivial systems & unknown physics

(stay tuned!)

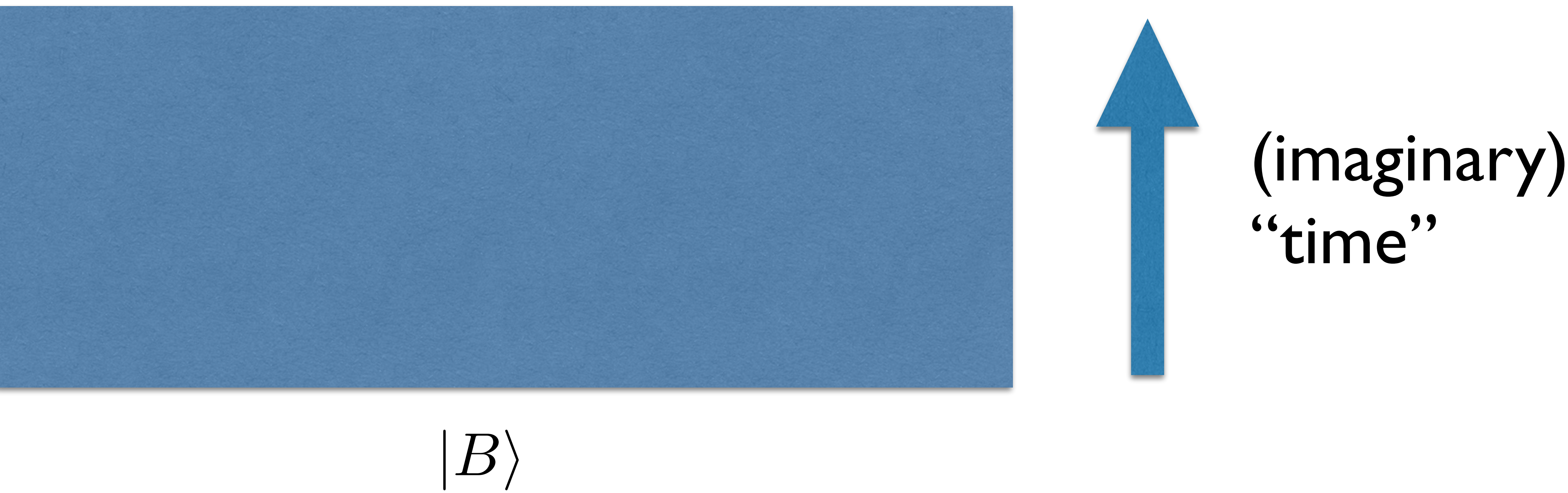
Boundary CFT

Critical system in 1+1D with a boundary:
invariance under conformal mapping which preserves the boundary: "real analytic function"

Still infinite dimensional symmetry
(Virasoro algebra)

For a given CFT, there are several conformally invariant boundary conditions, which correspond to fixed points of "boundary RG flow"

Boundary State



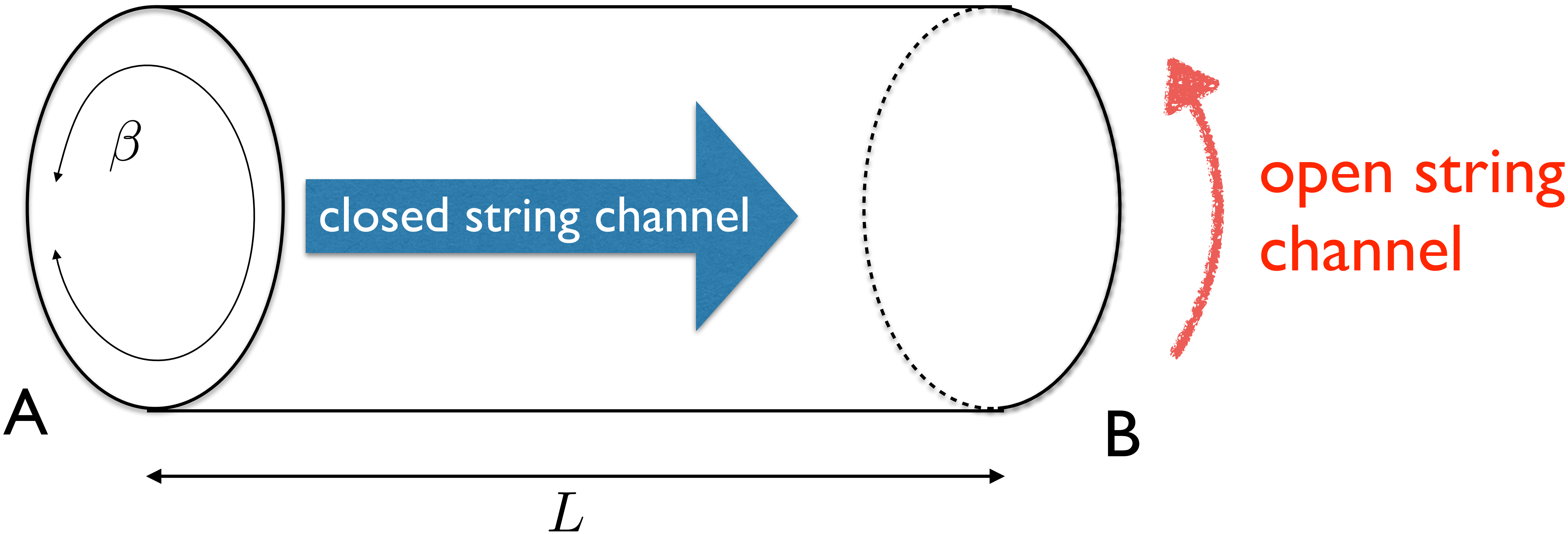
boundary condition ⇔ boundary state

$$\forall n : \quad (L_n - \bar{L}_n)|B\rangle = 0$$

$$|B\rangle = \sum_h c_B^h |h\rangle\rangle$$

$|h\rangle\rangle$ Ishibashi state for the primary field *h*

BCFT Partition Function



$$Z_{AB} = \langle B | e^{-L\mathcal{H}_\beta} | A \rangle = \sum_i (c_A^i)^* c_B^i \chi_i(\tilde{q}) \qquad \tilde{q} \equiv e^{-4\pi L/\beta}$$

modular transformation

$$Z_{AB} = \sum_j n_{AB}^j \chi_j(q) \qquad q \equiv e^{-\pi\beta/L}$$

Cardy Condition

$$|B\rangle = \sum_h c_B^h |h\rangle\rangle$$

$$Z_{AB} = \langle B| e^{-L\mathcal{H}_\beta} |A\rangle = \sum_i (c_A^i)^* c_B^i \chi_i(\tilde{q}) \qquad \tilde{q} \equiv e^{-4\pi L/\beta}$$

$\Updownarrow$ modular transformation

$$Z_{AB} = \sum_j n_{AB}^j \chi_j(q) \qquad q \equiv e^{-\pi\beta/L}$$

$n_{AB}^j \in \mathbb{Z}_{\geq 0}$ # of primary field j with the boundary conditions A & B

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Nuclear Physics B 570 [FS] (2000) 525–589

NUCLEAR PHYSICS B

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Boundary conditions in rational conformal field theories

Roger E. Behrend [a], Paul A. Pearce [b], Valentina B. Petkova [c,1], Jean-Bernard Zuber [d]

We develop further the theory of Rational Conformal Field Theories (RCFTs) on a cylinder with specified boundary conditions emphasizing the role of a triplet of algebras: the Verlinde, graph fusion and Pasquier algebras. We show that solving Cardy's equation, expressing consistency of a RCFT on a cylinder, is equivalent to finding integer valued matrix representations of the Verlinde algebra. These matrices allow us to naturally associate a graph G to each RCFT such that the conformal boundary conditions are labelled by the nodes of G. This approach is carried to completion for $sl(2)$ theories leading to complete sets of conformal boundary conditions, their associated cylinder partition functions and the A-D-E classification. We also review the current status for WZW $sl(3)$ theories. Finally, a systematic generalisation of the formalism of Cardy–Lewellen is developed to allow for multiplicities arising from more general representations of the Verlinde algebra. We obtain information on the bulk–boundary coefficients and reproduce the relevant algebraic structures from the sewing constraints.

Rational or Irrational?

Rational CFT:

CFT with a finite number of primary fields
(representation of Virasoro algebra)

$$\frac{\infty}{\infty} = \mathrm{finite}$$ reduced to a finite-dimensional problem

Irrational CFT:

CFT with an infinite number of primary fields

$$\frac{\infty}{\infty} = \infty$$ more difficult as a CFT

Classification of conformal b.c. : generally unknown

Free Boson: Irrational CFT!

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 \quad \longleftrightarrow \quad \mathcal{L} = \frac{K}{2\pi}(\partial_\mu \theta)^2$$

"*T* duality"

$$\phi \sim \phi + \pi \qquad\qquad \theta \sim \theta + 2\pi$$

K: "Luttinger parameter"

Infinite number of primary fields include

$$V_{n,m} \equiv e^{2in\phi + im\theta} \qquad n, m \in \mathbb{Z}$$

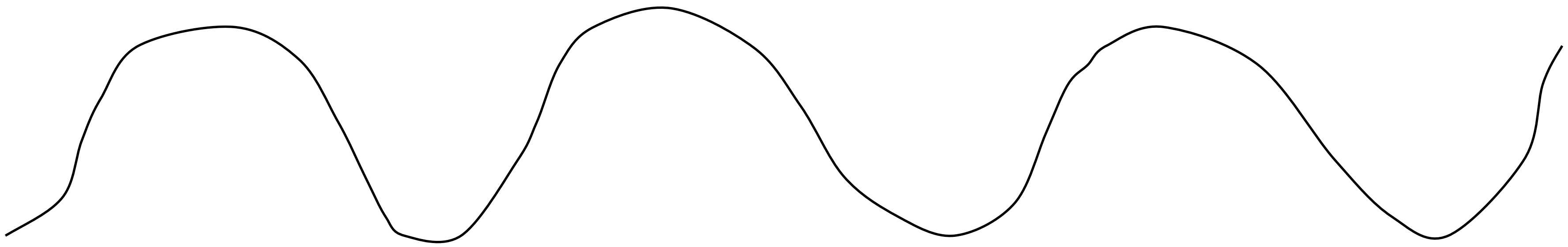
General classification of irrational CFT: unknown

Tomonaga-Luttinger Liquid

S. Tomonaga

朝永振一郎 (1906～1979)

In 1D, all the low-energy excitations in the interacting electron system can be described in terms of "phonons" (free bosons)



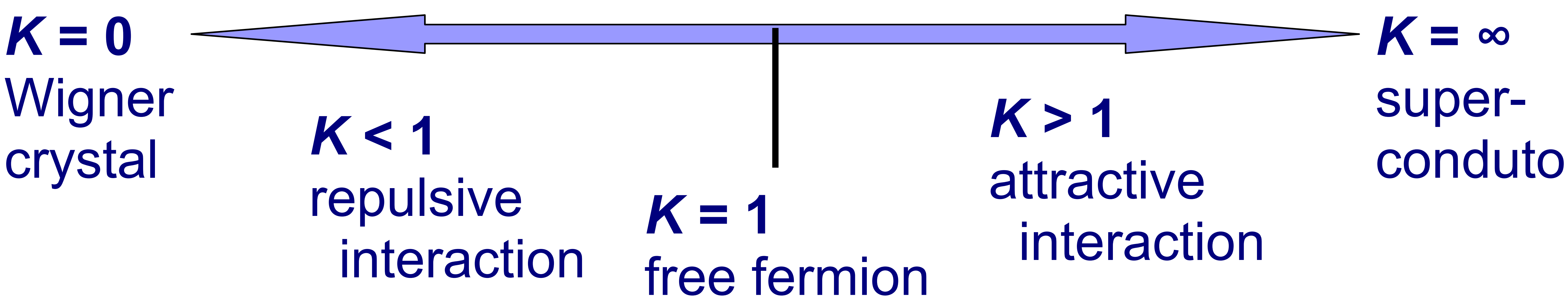
equivalent to (quantized) vibrating "string"

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\phi = \frac{\partial^2}{\partial x^2}\phi \quad \Longleftrightarrow \quad \mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2$$

Tomonaga-Luttinger Liquid

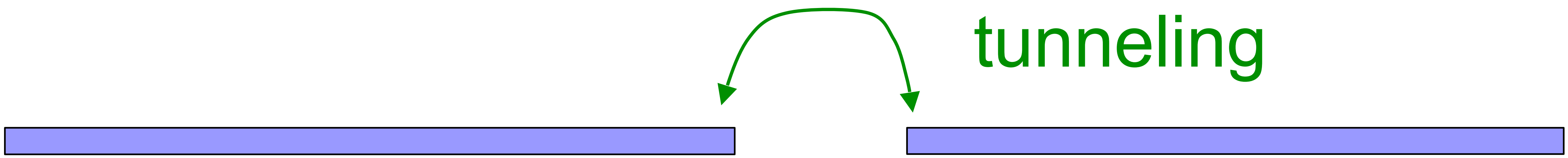
The free boson theory appears trivial---
but it contains rich physics!

Luttinger parameter K represents the strength of quantum fluctuation, and reflects the electron interactions



Junction of 2 quantum wires

Kane-Fisher (1992), Furusaki-Nagaosa (1993)



For non-interacting electrons:

transmission probability:
continuous function of the tunneling amplitude

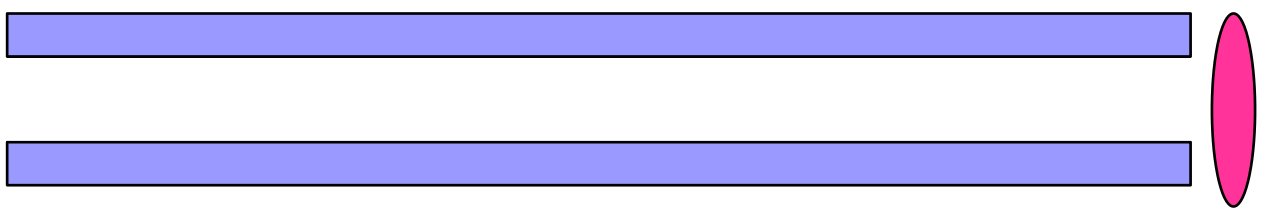
conductance also depends continuously
on the tunneling amplitude

However, we will see that the interaction fundamentally changes the physics!

Field theory on the junction

“fold” the system at the junction

two-component boson field theory



charge conservation at the junction
“kills” the total charge mode

boundary problem of a single-component free boson field theory (c=1 CFT)

Boundary condition of free boson

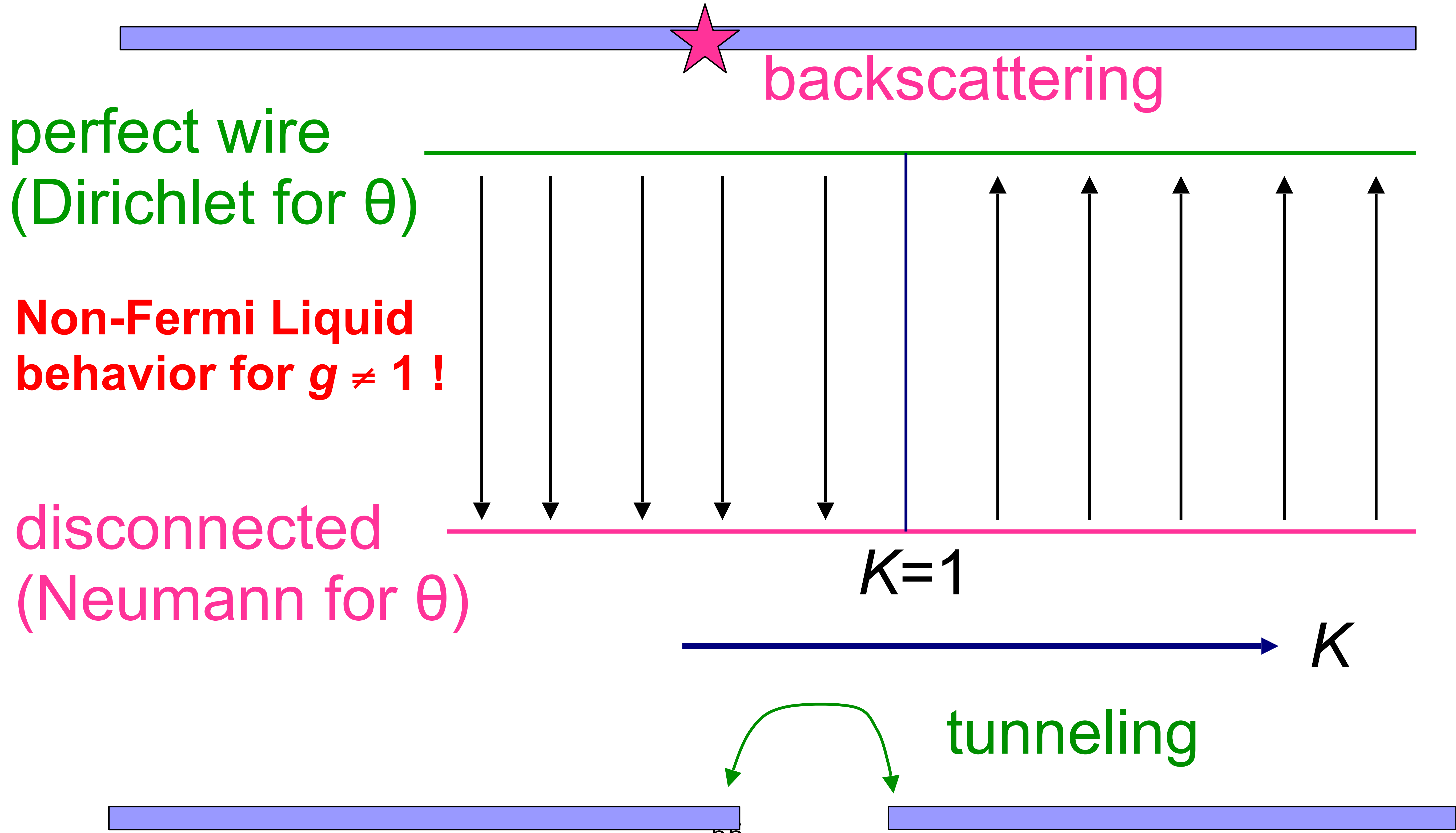
In the low-energy limit, the boundary condition approaches to a conformally invariant boundary condition

Known conformally invariant boundary conditions:

Neumann b.c. ⟺ Disconnected wires

Dirichlet b.c. ⟺ Perfect transmission (merged to a single wire)

Phase diagram for 2 wires



Complete Set of BC?

From the RG analysis, there was no indication that nontrivial conformally invariant b.c. exists

The conformally invariant boundary conditions for the c=1 free boson CFT are exhausted by Dirichlet, Neumann (and their SU(2) rotation?)

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Exceptional boundary states at $c = 1$

Romuald A. Janik
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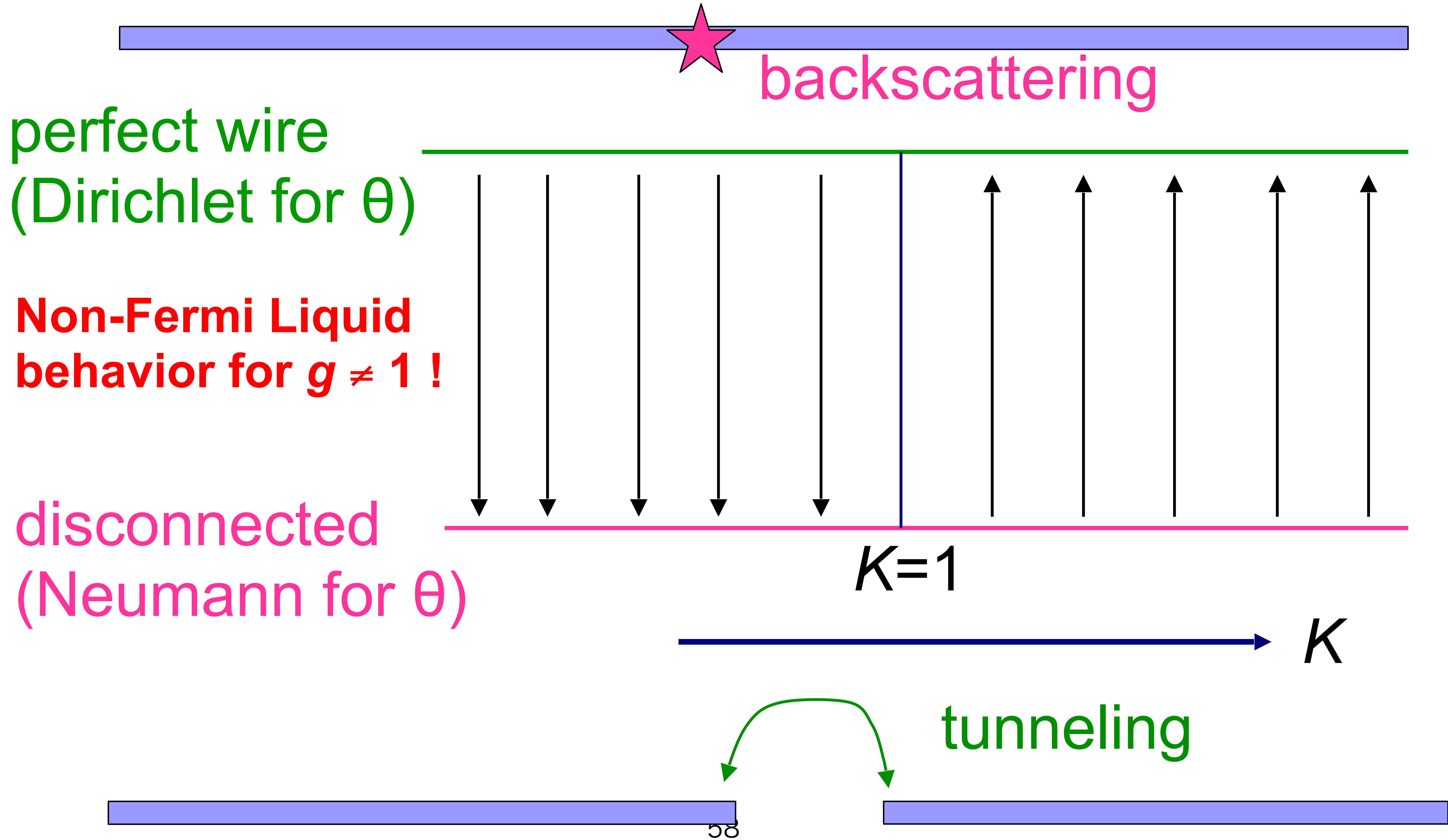
[1] D. Friedan, The space of conformal boundary conditions for the $c = 1$ Gaussian model, unpublished note, 1999.

When we are in the rational setting the sum is finite and the coefficients have to be integers (this is the main content of Cardy's condition). In the nonrational case the summation is transformed into an integral and the meaning of Cardy's condition then is not completely understood, in particular, it is not obvious what conditions one should impose on the $n^A_{\alpha\beta}$ which now is a 'density function' (measure) on the space of conformal weights A.

The least that we may do is to require that for *any* choice of boundary states $|\alpha\rangle$, $|\beta\rangle$ the density function $n^A_{\alpha\beta}$ should be *non-negative*. In Section 6 we will verify that this property holds for our construction of Friedan's boundary states.

Is this really physical? (Dirichlet/Neumann has non-negative integer coefficients)

Phase diagram for 2 wires



PHYSICAL REVIEW B **85**, 045120 (2012)

General method for calculating the universal conductance of strongly correlated junctions of multiple quantum wires

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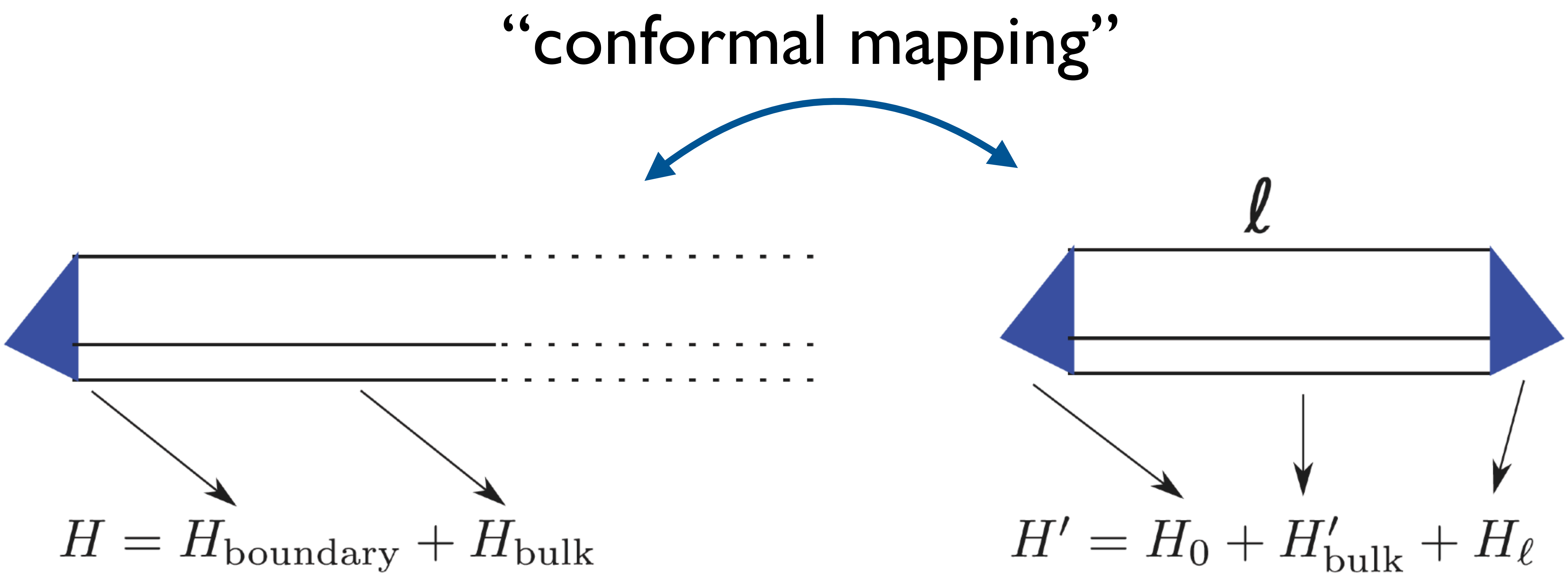
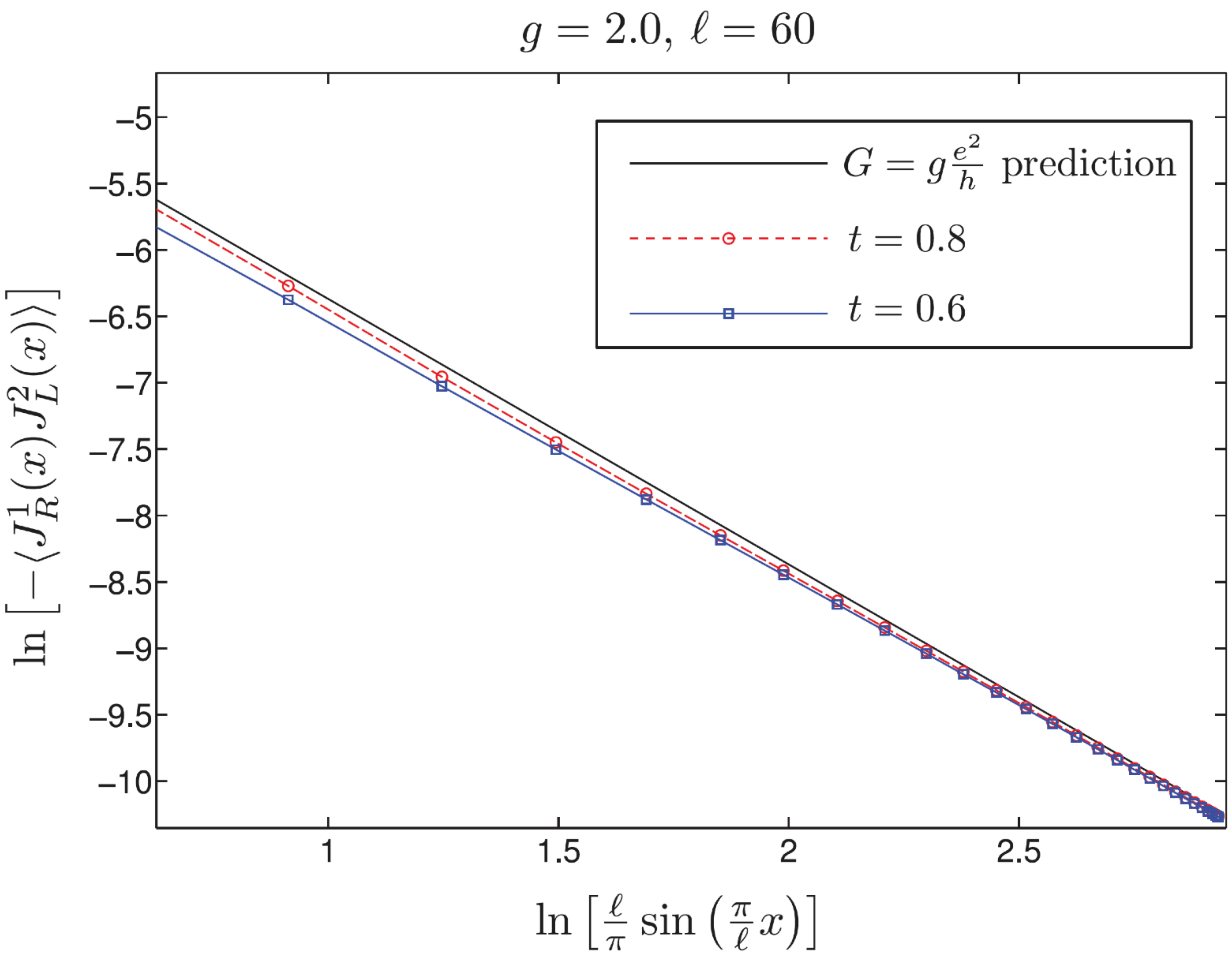
(Received 22 August 2011; revised manuscript received 28 November 2011; published 20 January 2012)




$g = 2.0,\ \ell = 60$
$G = g\frac{e^2}{h}$ prediction
$t = 0.8$
$t = 0.6$
$\ln\left[-\langle J_R^1(x) J_L^2(x)\rangle\right]$
$\ln\left[\frac{\ell}{\pi}\sin\left(\frac{\pi}{\ell}x\right)\right]$
-5
-5.5
-6
-6.5
-7
-7.5
-8
-8.5
-9
-9.5
-10
1
1.5
2
2.5

Issues

Effective system size = "max chord distance" $\frac{l}{\pi}$

Need to construct the "mirror image" of the boundary (junction) carefully

At short lengthscale, the boundary is not yet conformally invariant (boundary RG flow)

Crossover of correlation functions (UV → IR) cannot be studied faithfully which are based on the conformal mapping

Calculation without the conformal mapping?

PHYSICAL REVIEW B **99**, 121103(R) (2019)

Rapid Communications

Crossover of correlation functions near a quantum impurity in a Tomonaga-Luttinger liquid

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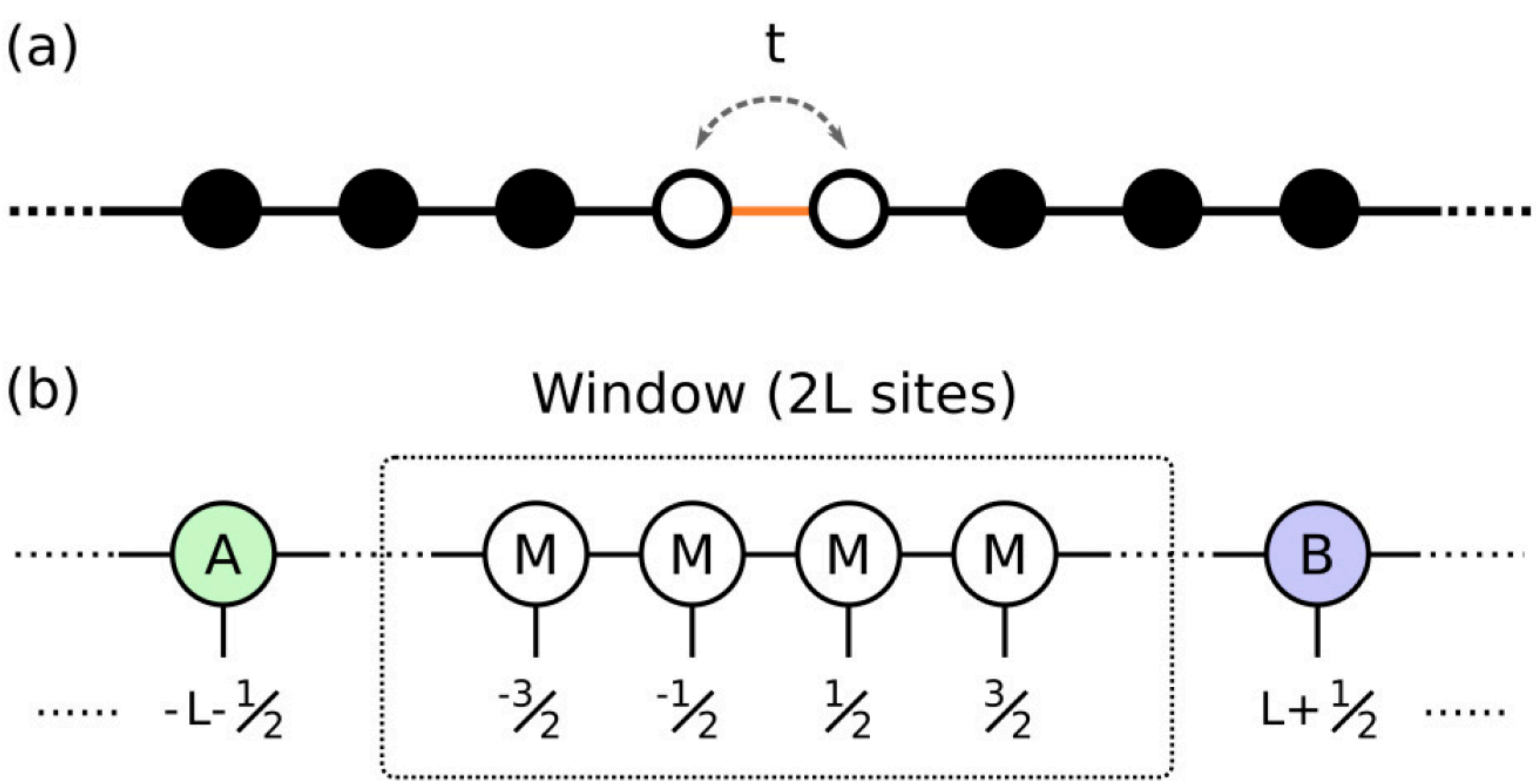
(Received 24 May 2018; revised manuscript received 22 October 2018; published 7 March 2019)



infinite DMRG
+ finite window

Window size

$2L \leqq 400$

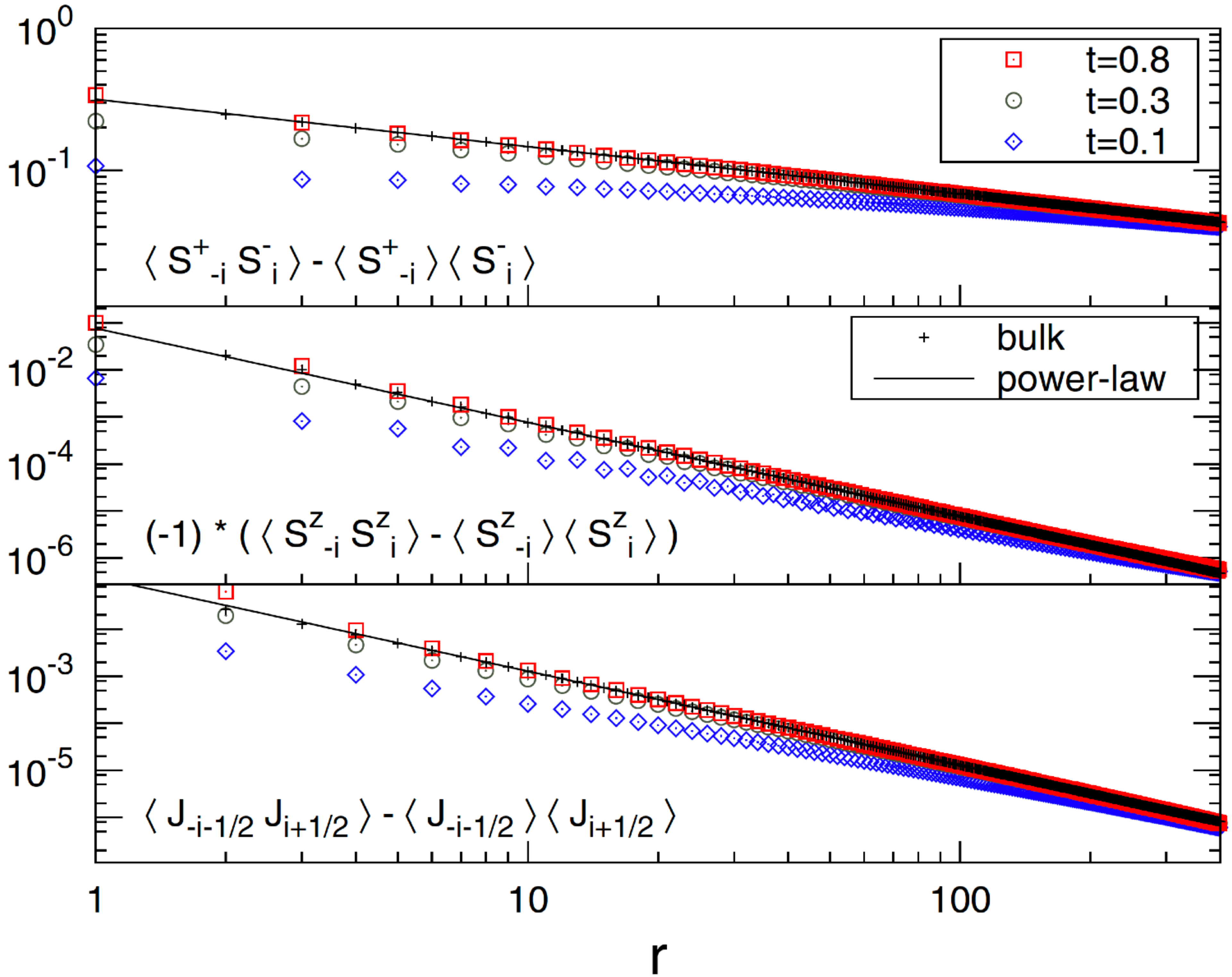


FIG. 2. $\langle S^+_{-i} S^-_i \rangle$, $\langle S^z_{-i} S^z_i \rangle$, and $\langle J_{-i-1/2} J_{i+1/2} \rangle$ correlation functions for $g = 1.5$. Data for the bulk and junctions with $t = 0.1, 0.3$, and 0.8 are plotted. Solid lines are power-law fittings to the bulk data with bulk exponents from bosonization (cf. Table I).

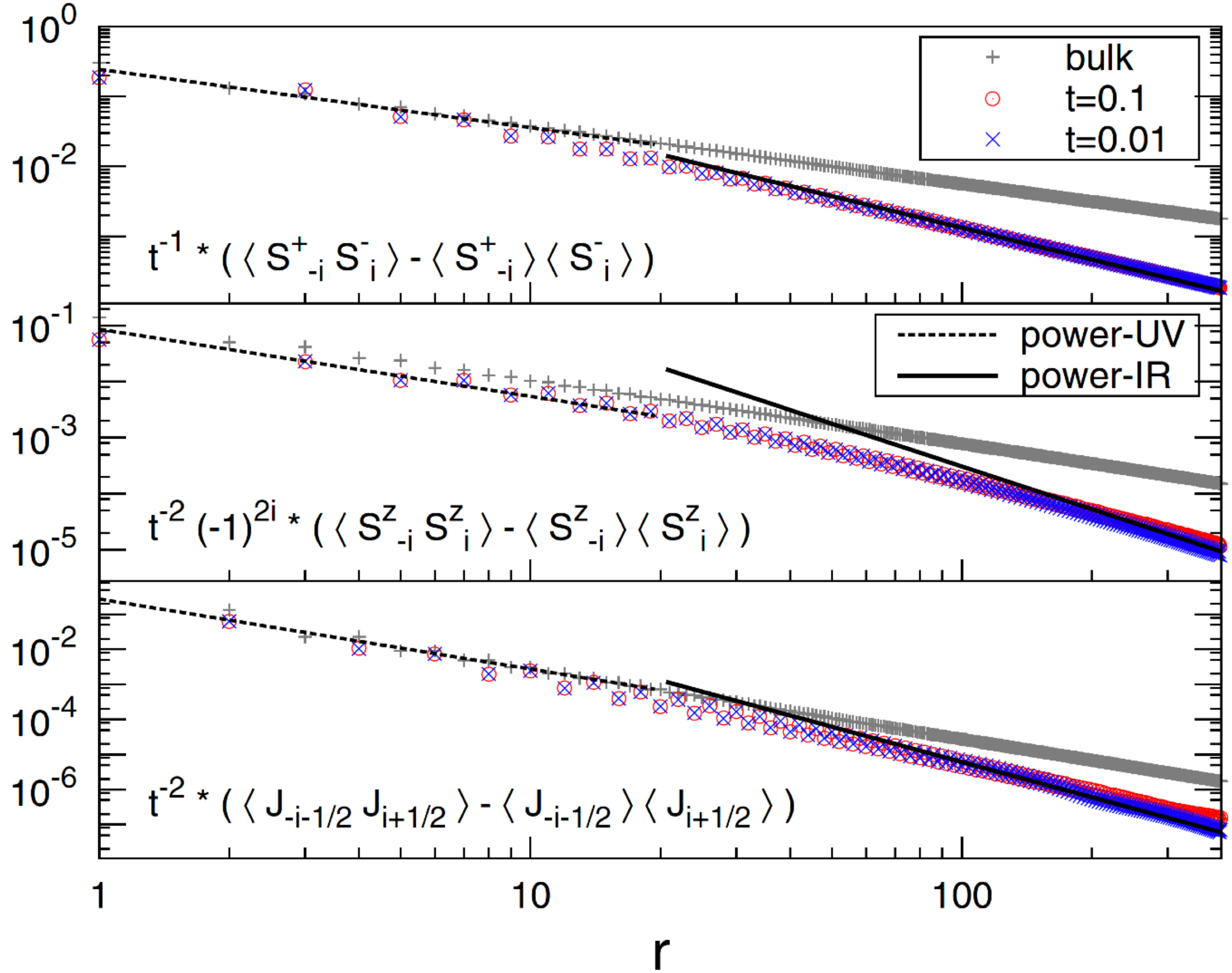
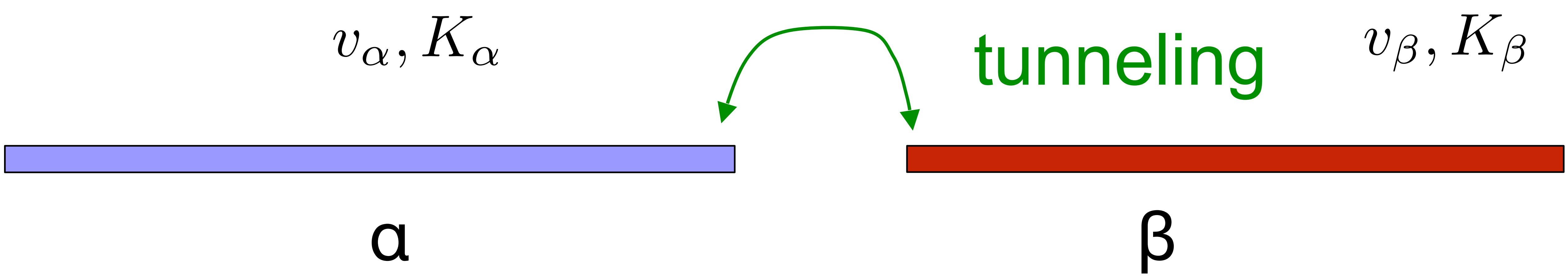


FIG. 3. Rescaled $\langle S^{+}_{-i}S^{-}_{i}\rangle$, $\langle S^{z}_{-i}S^{z}_{i}\rangle$, and $\langle J_{-i-1/2}J_{i+1/2}\rangle$ correlation functions for $g = 0.6$. Data for the bulk and junctions with $t = 0.1$ and 0.01 are plotted. Solid (dotted) lines are power-law fittings to the long- (short-) distance data with IR (UV) exponents from bosonization, except that for $\langle S^{+}_{-i}S^{-}_{i}\rangle$ at the IR limit the prefactor $C_0 = 1.330$ from bosonization is used [32] (cf. Table I).

Junction of 2 Inequivalent TLLs



Boundary CFT by rescaling

$$z_\mu = v_\mu \tau + i x \quad \Rightarrow \quad \langle J^\alpha(y,0) J^\beta(x,0) \rangle = -\frac{G}{2\pi^2} \frac{v^\alpha v^\beta}{(x+y)^2}$$

$$z_\mu = \tau + i \frac{x}{v_\mu} \quad \Rightarrow \quad \langle J^\alpha(y,0) J^\beta(x,0) \rangle = -\frac{G}{2\pi^2} \frac{1}{(\frac{x}{v_\alpha} + \frac{y}{v_\beta})^2}$$

Junction of 2 Inequivalent TLLs

$$\frac{1}{g_e} = \frac{1}{2}\left(\frac{1}{g^\alpha} + \frac{1}{g^\beta}\right)$$

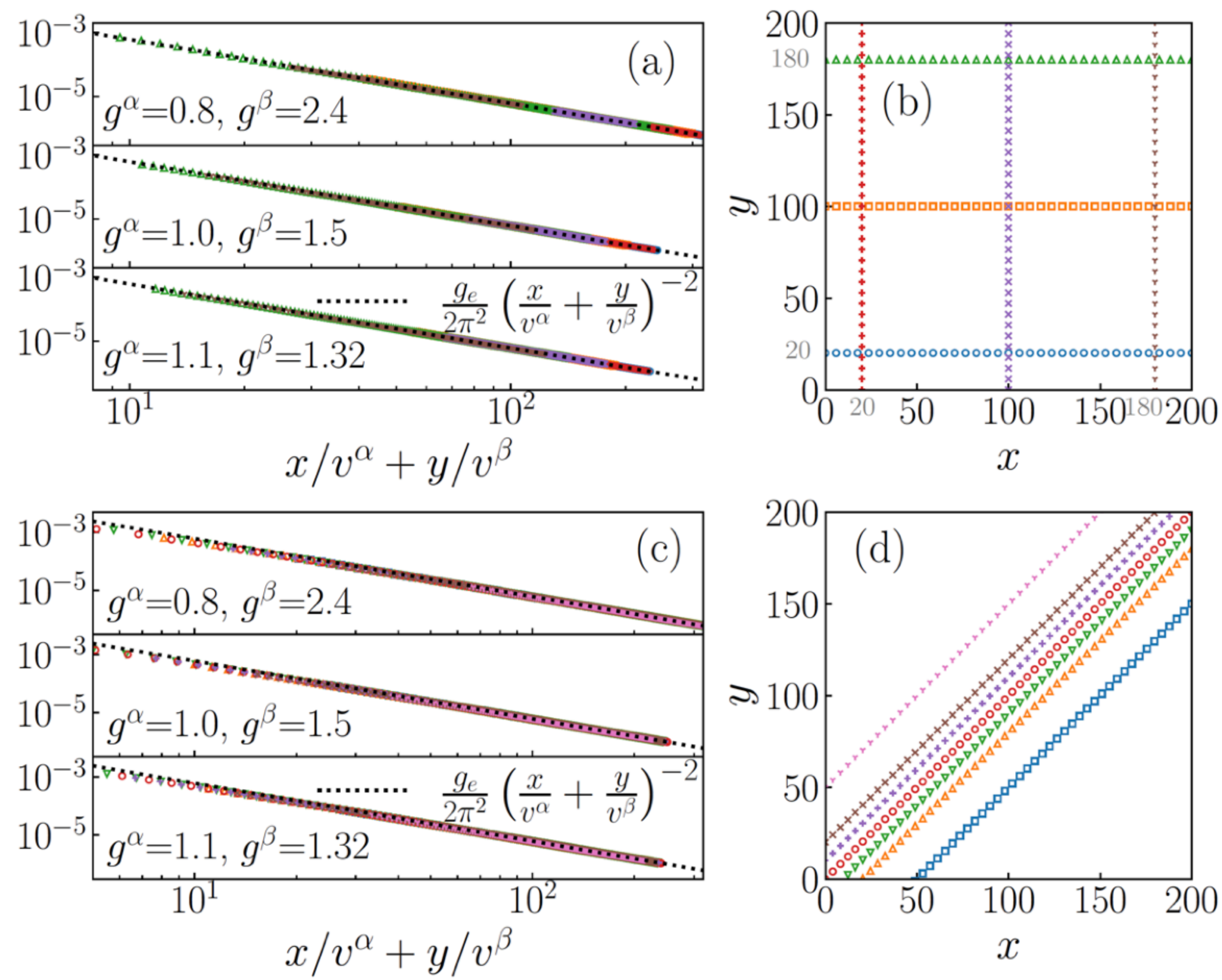
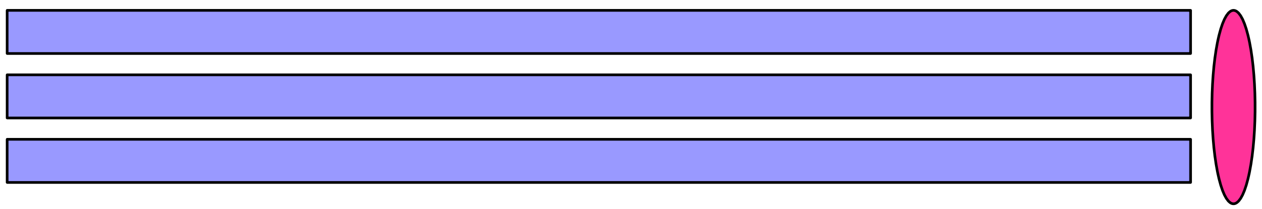


FIG. 2. (a) and (c) are equal-time current-current correlation functions $\langle J^\alpha(x)J^\beta(y)\rangle$ versus $x/v^\alpha + y/v^\beta$, where x and y are limited to the dotted lines in (b) and (d) respectively.

Field theory on a 3-wires junction

"fold" the system at the junction

3-component boson field theory



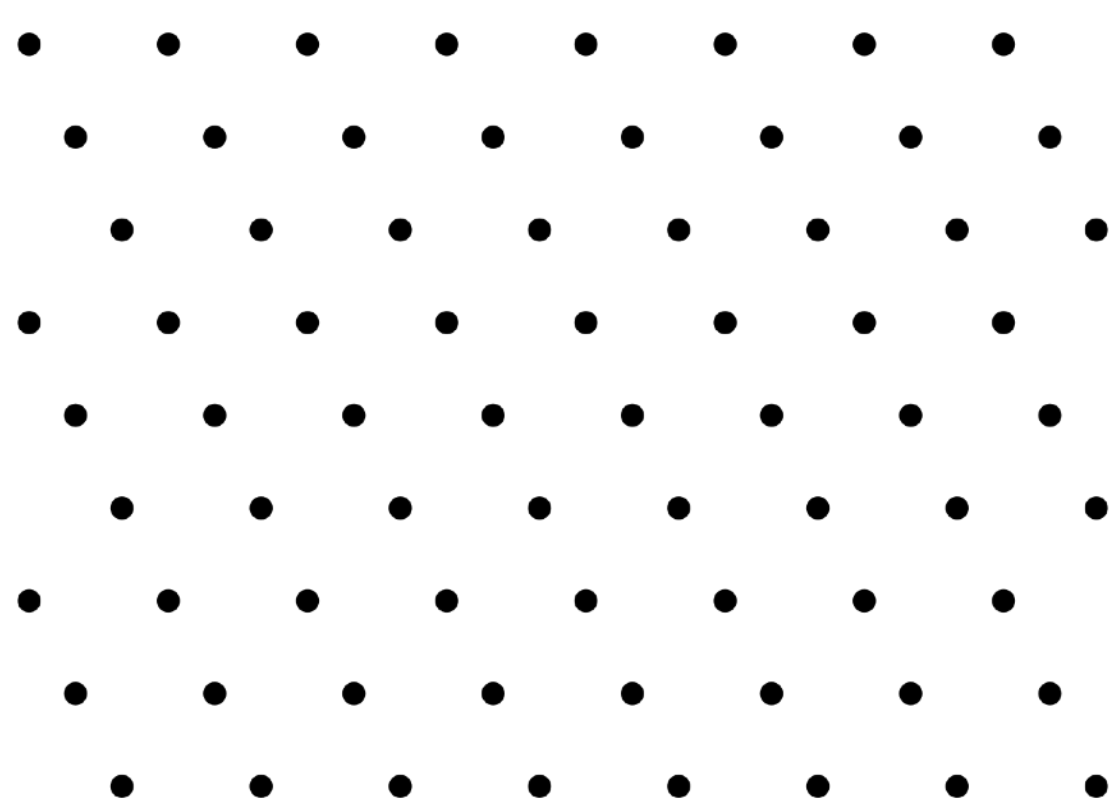
charge conservation at the junction
"kills" the total charge mode

boundary problem of a 2-component
free boson field theory ($c=2$ CFT) of $\vec{X}$

Non-trivial Boundary Conditions

For c ≧ 2 free boson CFT,

there are nontrivial conformally invariant boundary conditions (satisfying the original Cardy condition) other than Neumann/Dirichlet!



oblique compactification lattice for *c*=2

$$\vec{\theta} \sim \vec{\theta} + 2\pi \vec{R}$$

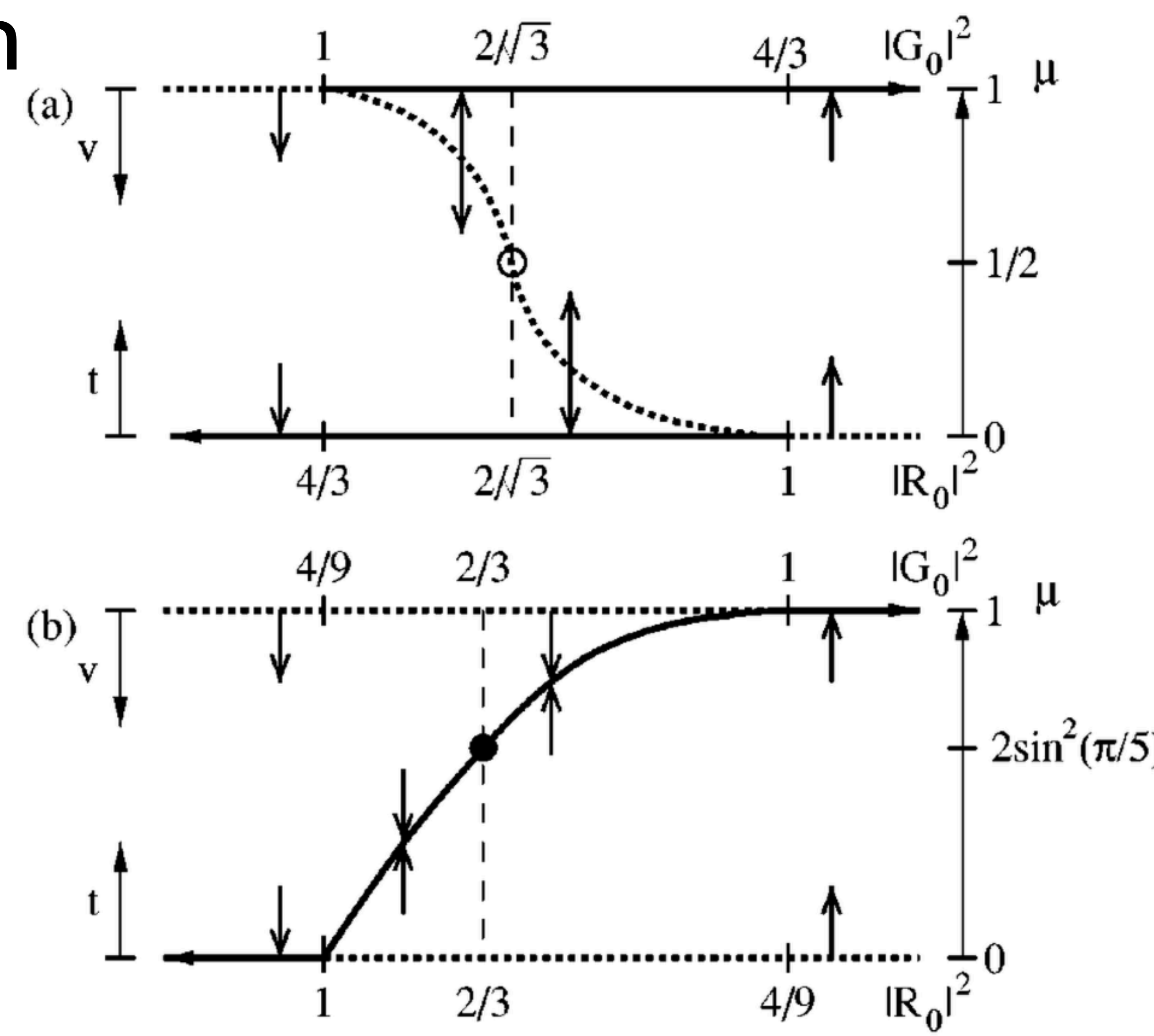
$$\vec{R} \in \Lambda$$

RG argument

Necessity from the RG argument

Affleck-Ludwig 1991, Furusaki-Nagosa 1993, Yi-Kane 1997 etc.

when the "compactification lattice" is oblique, Dirichlet & Neumann can be stable or unstable simultaneously



intermediate RG fixed point = nontrivial conformally invariant boundary cond.

Fusion Construction

Given a consistent conformal b.c. A $\quad Z_{AA} = \sum_i n^i_{AA}\chi_i(q)$

new consistent b.c. B can be constructed by fusion

$$c^i_B = c^i_A \frac{S^i_k}{S^i_0}$$

fusion with primary field *k* using modular *S*-matrix

$$\chi_i(\tilde{q}) = \sum_j S^j_i \chi_j(q)$$

$$n^i_{AB} = \sum_j N^i_{jk} n^j_{AA}$$

Verlinde formula

$$\sum_j S^i_j N^j_{kl} = \frac{S^i_k S^i_l}{S^i_0}$$

$N^i_{jk} \in \mathbb{Z}_{\geq 0}$ fusion rule coefficient for OPE

Nontrival c=2 Partition Function

c=2 free boson CFT with
a special compactification lattice
admits conformal embedding

Affleck-M.O.-Saleur 2000

$$2 = \frac{1}{2} + \frac{7}{10} + \frac{4}{5}$$

Ising + tricritical Ising + Potts

$$\begin{aligned} Z_{DD}(q) &= (\chi_0^I\chi_0^T + \chi_{1/2}^I\chi_{3/2}^T)(\chi_0^P + \chi_3^P) \\ &+ (\chi_0^I\chi_{3/5}^T + \chi_{1/2}^I\chi_{1/10}^T)(\chi_{2/5}^P + \chi_{7/5}^P), \\ Z_{NN}(q) &= (\chi_0^I\chi_0^T + \chi_{1/2}^I\chi_{3/2}^T)(\chi_0^P + \chi_3^P + 2\chi_{2/3}^P) \\ &+ (\chi_0^I\chi_{3/5}^T + \chi_{1/2}^I\chi_{1/10}^T)(2\chi_{1/15}^P + \chi_{2/5}^P + \chi_{7/5}^P), \end{aligned}$$

Alternative Embedding

Affleck-M.O.-Saleur 2000

$$2 = \frac{6}{5} + \frac{4}{5}$$

$Z_3^{(5)}$ + (Potts = $Z_3^{(4)}$)

$$\begin{aligned} Z_{DD}(q) =& (\chi_0^5 + 2\chi_2^5)(\chi_0^P + \chi_3^P) \\ &+ (2\chi_{3/5}^5 + \chi_{8/5}^5)(\chi_{2/5}^P + \chi_{7/5}^P) \\ Z_{NN}(q) =& (\chi_0^5 + 2\chi_2^5)(\chi_0^P + \chi_3^P + 2\chi_{2/3}^P) \\ &+ (2\chi_{3/5}^5 + \chi_{8/5}^5)(2\chi_{1/15}^P + \chi_{2/5}^P + \chi_{7/5}^P) \end{aligned}$$

In either embedding, we can apply fusion construction in each sector to possibly obtain new boundary states

Boundary state	g-factor	μ_{xx}	μ_{yy}
D	$1/\sqrt{2\sqrt{3}}$	0	0
N	$\sqrt{\sqrt{3}/2}$	1	1
Y	$2\cos(\pi/5)/\sqrt{2\sqrt{3}}$	$(5-\sqrt{5})/4$	$(5-\sqrt{5})/4$
W	$2\cos(\pi/5)\sqrt{\sqrt{3}/2}$	$(-1+\sqrt{5})/4$	$(-1+\sqrt{5})/4$
R	$[3(7+3\sqrt{5})/2]^{1/4}$	$(-1+\sqrt{5})/4$	$(5-\sqrt{5})/4$
S	$3^{1/4}$	1	0
T	$(7+3\sqrt{5}/6)^{1/4}$	$(5-\sqrt{5})/4$	$(-1+\sqrt{5})/4$
U	$1/3^{1/4}$	0	1

Boundary state	g-factor	μ_{xx}	μ_{xy}
F	$\sqrt{\frac{2}{\sqrt{3}}}$	$\frac{3}{4}$	$\pm\frac{\sqrt{3}}{4}$
X	$[\frac{14}{3}+2\sqrt{5}]^{1/4}$	$\frac{1+\sqrt{5}}{8}$	$\pm\frac{3\sqrt{3}-\sqrt{15}}{8}$
V	$[6(7+3\sqrt{5})]^{1/4}$	$\frac{7-\sqrt{5}}{8}$	$\pm\frac{3\sqrt{3}-\sqrt{15}}{8}$
Z	$\sqrt{2\sqrt{3}}$	$\frac{1}{4}$	$\pm\frac{\sqrt{3}}{4}$

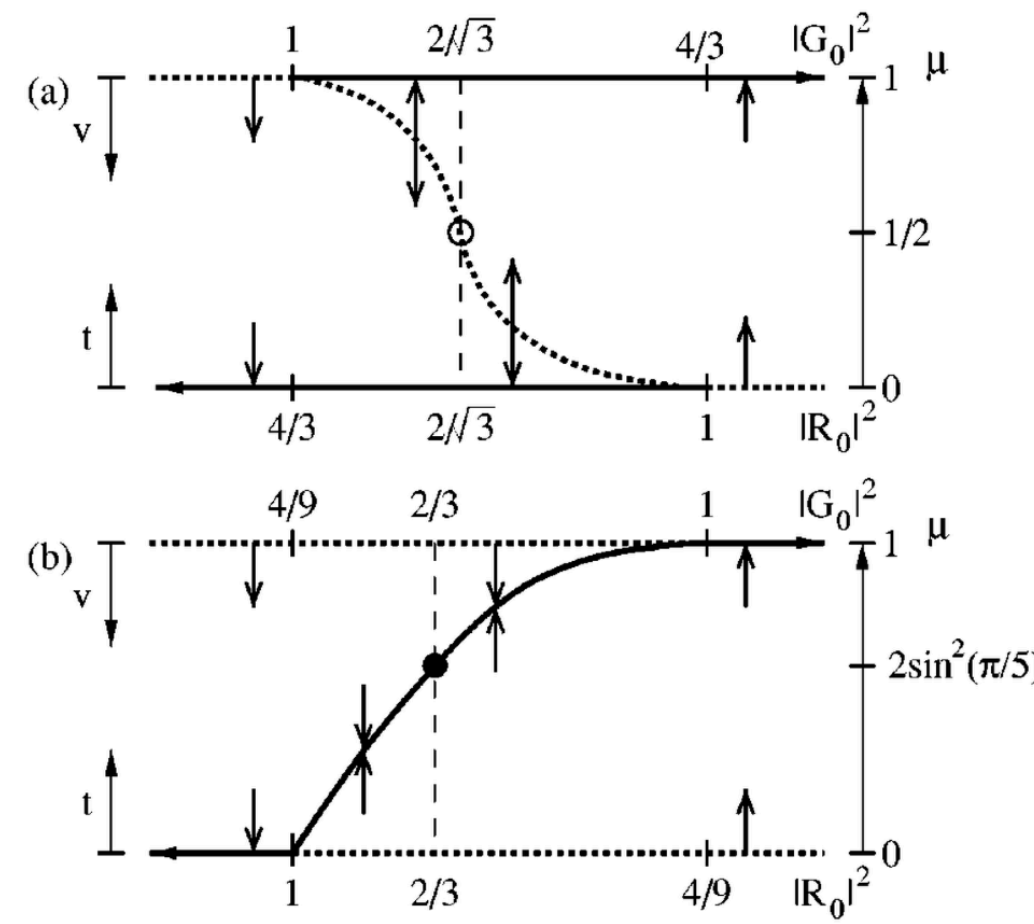
Affleck-M.O.-Saleur 2000

“Zoo” of Conformal B.C.

Presumably incomplete

In fact, the conformal embedding works only for a special compactification (Luttinger parameter) but the similar conformal boundary condition should exist for a range of parameters

Classification of conformally invariant boundary conditions for irrational CFTs?



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Nuclear Physics B 941 (2019) 794–837

NUCLEAR PHYSICS B

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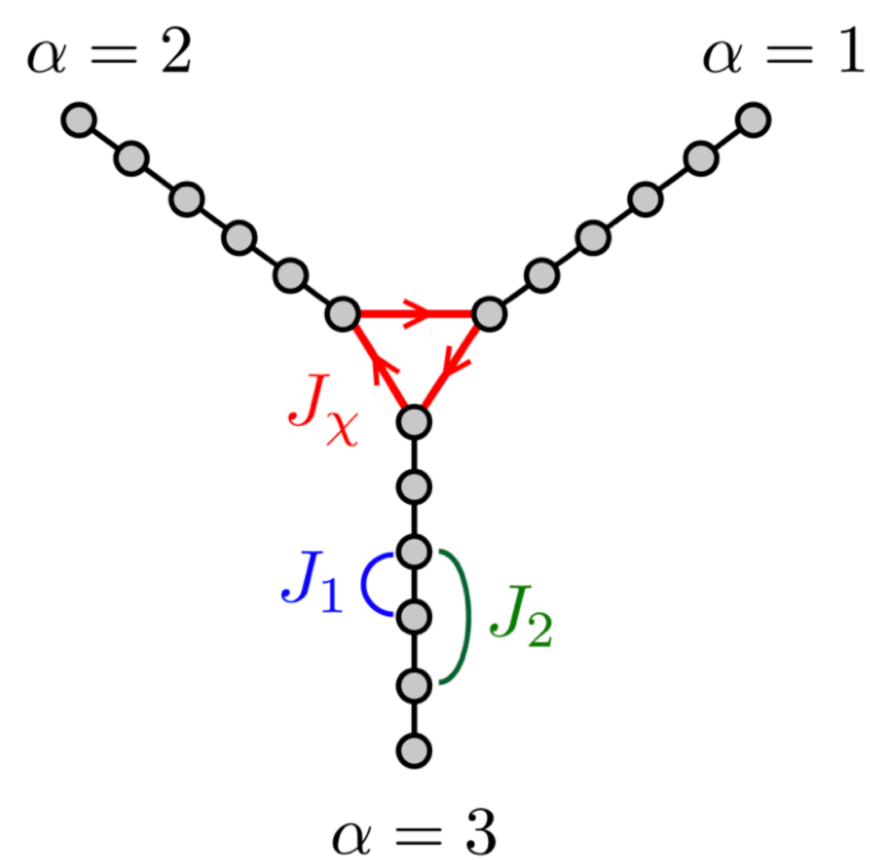
Chiral Y junction of quantum spin chains

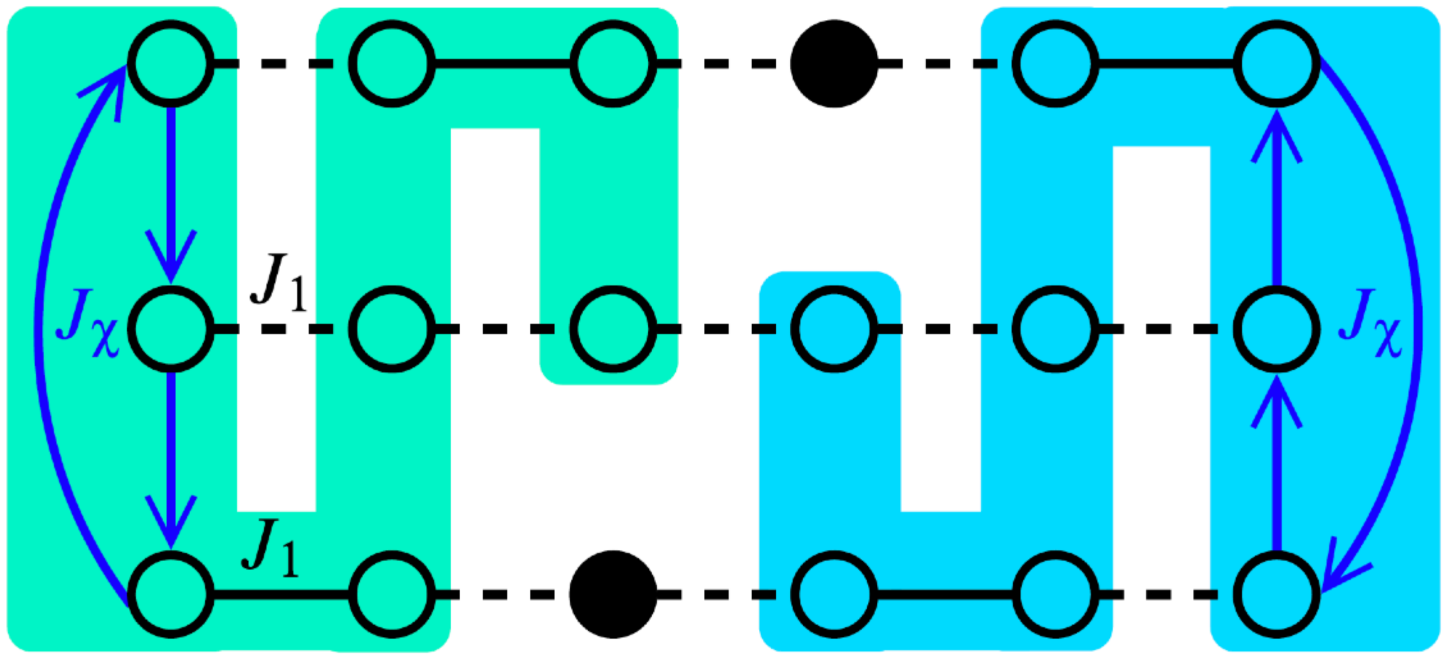
F. Buccheri [a,*], R. Egger [a], R. G. Pereira [a,b,c], F. B. Ramos [b]

$$3 = \frac{9}{5} + \frac{6}{5}$$

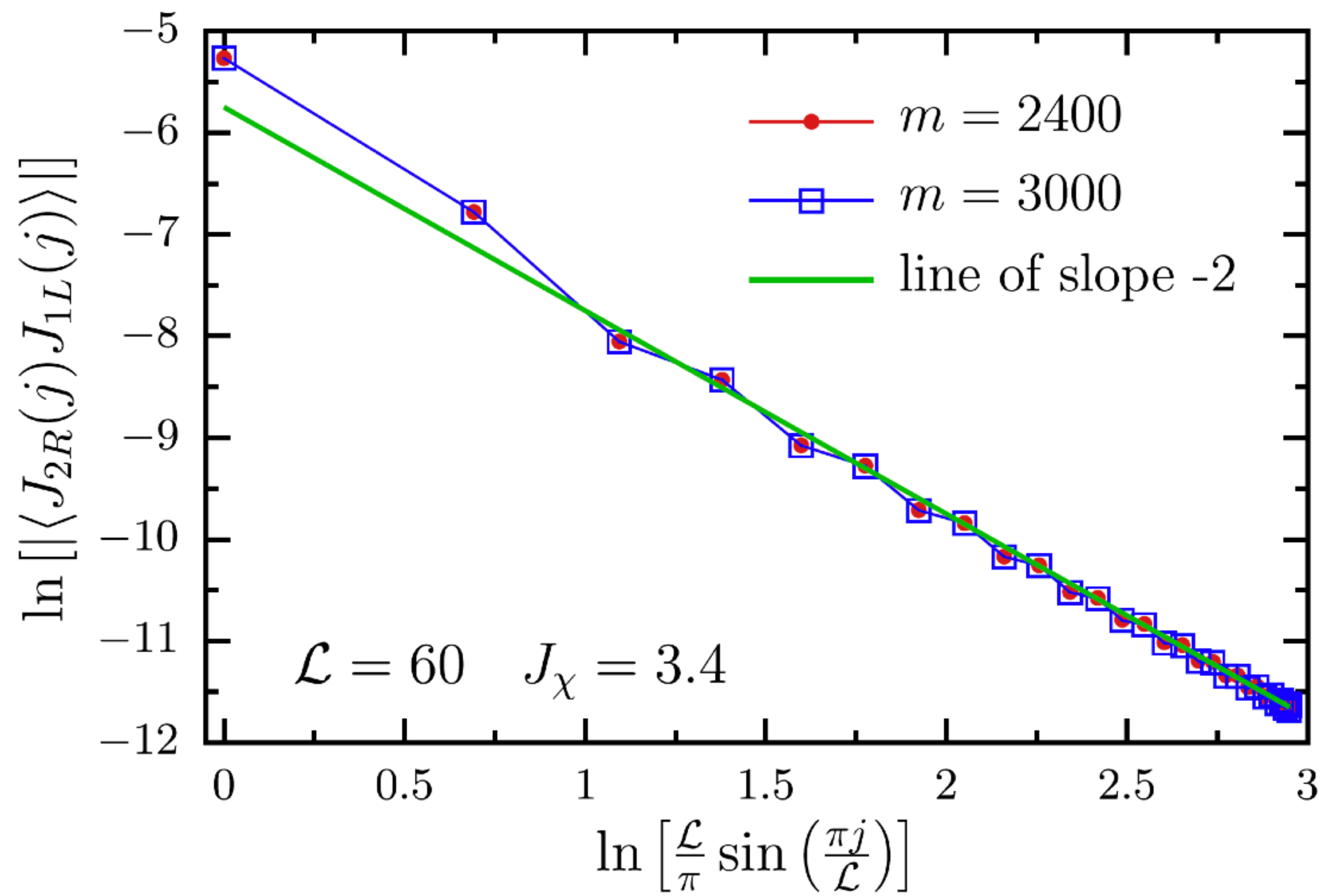
$$\mathrm{SU}(2)_3 \times \mathbb{Z}_3^{(5)}$$

$$3 = 2 + 1 = 1 + \frac{4}{5} + \frac{6}{5}$$





J_χ
J_1
J_1
J_χ



$m = 2400$
$m = 3000$
line of slope -2
$\mathcal{L} = 60 \quad J_\chi = 3.4$
$\ln[|\langle J_{2R}(j) J_{1L}(j)\rangle|]$
$\ln\left[\frac{\mathcal{L}}{\pi}\sin\left(\frac{\pi j}{\mathcal{L}}\right)\right]$

Table 1
Comparison of the (absolute value of the) dimensionless spin conductance G_{12}/G_0 for three values of J_χ corresponding to the O, C and K points, respectively. The DMRG estimates for $J_\chi = 0.4$ and 3.4 were obtained for $\mathcal{L} = 68$. The value for $J_\chi = 10$ has been extrapolated to infinite size, as explained in the main text.

Fixed point	J_χ	DMRG	BCFT	Rel. error
O	0.4	0.004	0	
C	3.4	0.498	0.5	0.4%
K	10	0.217	0.2303 …	5.7%

"K" = Y (3-channel Kondo, Yi-Kane)
"C" = F (Chiral)

Boundary state	g-factor	μ_{xx}	μ_{yy}
D	$1/\sqrt{2\sqrt{3}}$	0	0
N	$\sqrt{\sqrt{3}/2}$	1	1
Y	$2\cos(\pi/5)/\sqrt{2\sqrt{3}}$	$(5-\sqrt{5})/4$	$(5-\sqrt{5})/4$
W	$2\cos(\pi/5)\sqrt{\sqrt{3}/2}$	$(-1+\sqrt{5})/4$	$(-1+\sqrt{5})/4$
R	$[3(7+3\sqrt{5})/2]^{1/4}$	$(-1+\sqrt{5})/4$	$(5-\sqrt{5})/4$

Boundary state	g-factor	μ_{xx}	μ_{xy}
F	$\sqrt{\frac{2}{\sqrt{3}}}$	$\frac{3}{4}$	$\pm\frac{\sqrt{3}}{4}$
X	$[\frac{14}{3}+2\sqrt{5}]^{1/4}$	$\frac{1+\sqrt{5}}{8}$	$\pm\frac{3\sqrt{3}-\sqrt{15}}{8}$
V	$[6(7+3\sqrt{5})]^{1/4}$	$\frac{7-\sqrt{5}}{8}$	$\pm\frac{3\sqrt{3}-\sqrt{15}}{8}$
Z	$\sqrt{2\sqrt{3}}$	$\frac{1}{4}$	$\pm\frac{\sqrt{3}}{4}$

Where's the Fermi statistics?

electrons are fermions !

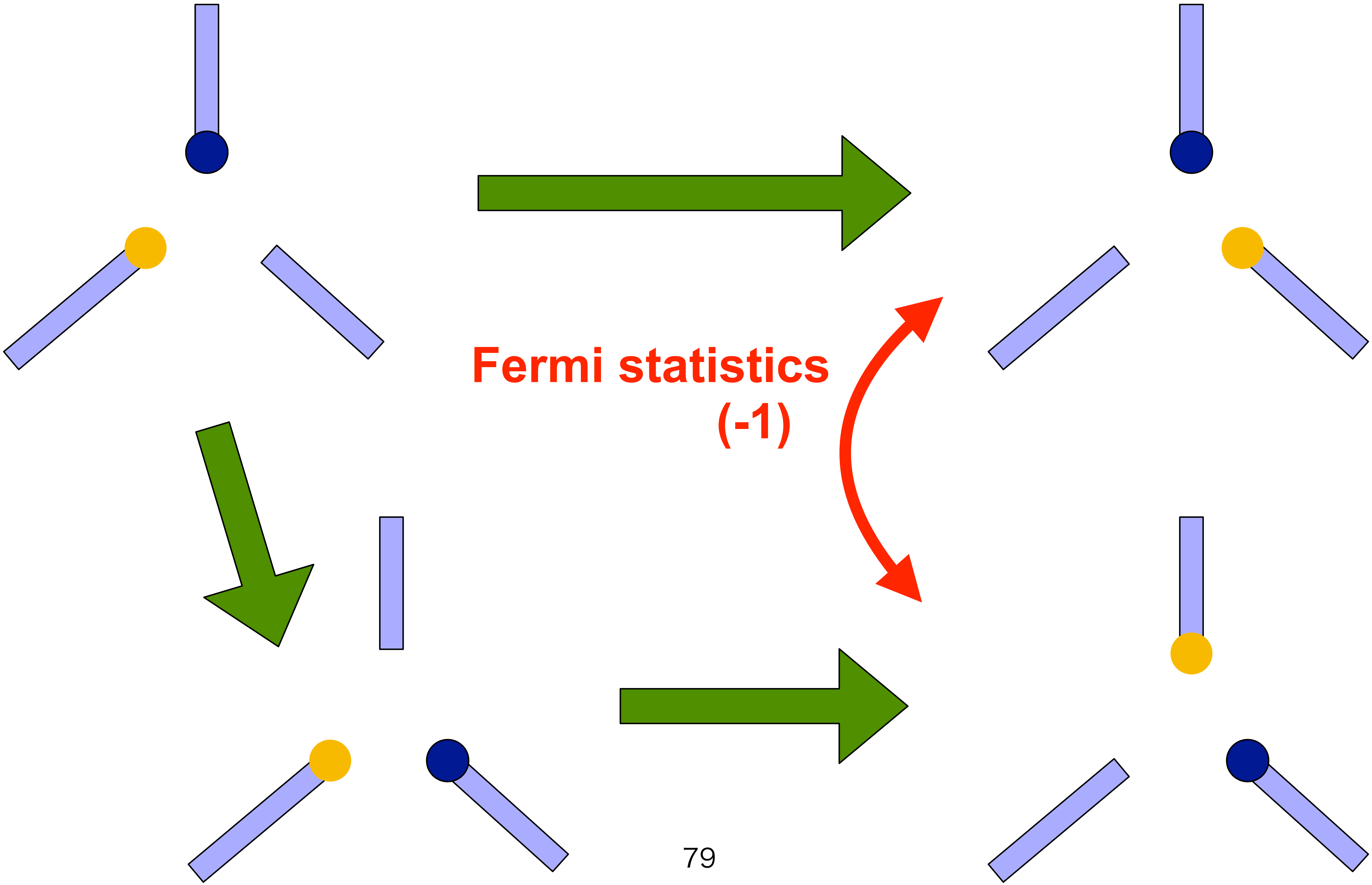
can we really reduce the junction to the problem of the free boson?

Answer: **YES** for the junction of 2 wires

(same result for the hard-core boson)

However, the Fermi statistics does play a crucial role in the junction of 3 wires

Fermi statistics is essential!



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Junctions of Three Quantum Wires and the Dissipative Hofstadter Model

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We study a junction of three quantum wires enclosing a magnetic flux. This is the simplest problem of a quantum junction between Tomonaga-Luttinger liquids in which Fermi statistics enter in a nontrivial way. We present a direct connection between this problem and the dissipative Hofstadter problem, or quantum Brownian motion in two dimensions in a periodic potential and an external magnetic field, which in turn is connected to open string theory in a background electromagnetic field. We find nontrivial fixed points corresponding to a chiral conductance tensor leading to an asymmetric flow of the current.

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Junctions of three quantum wires

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Phase diagram of the dissipative Hofstadter model

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Magnetic fields and fractional statistics in boundary conformal field theory

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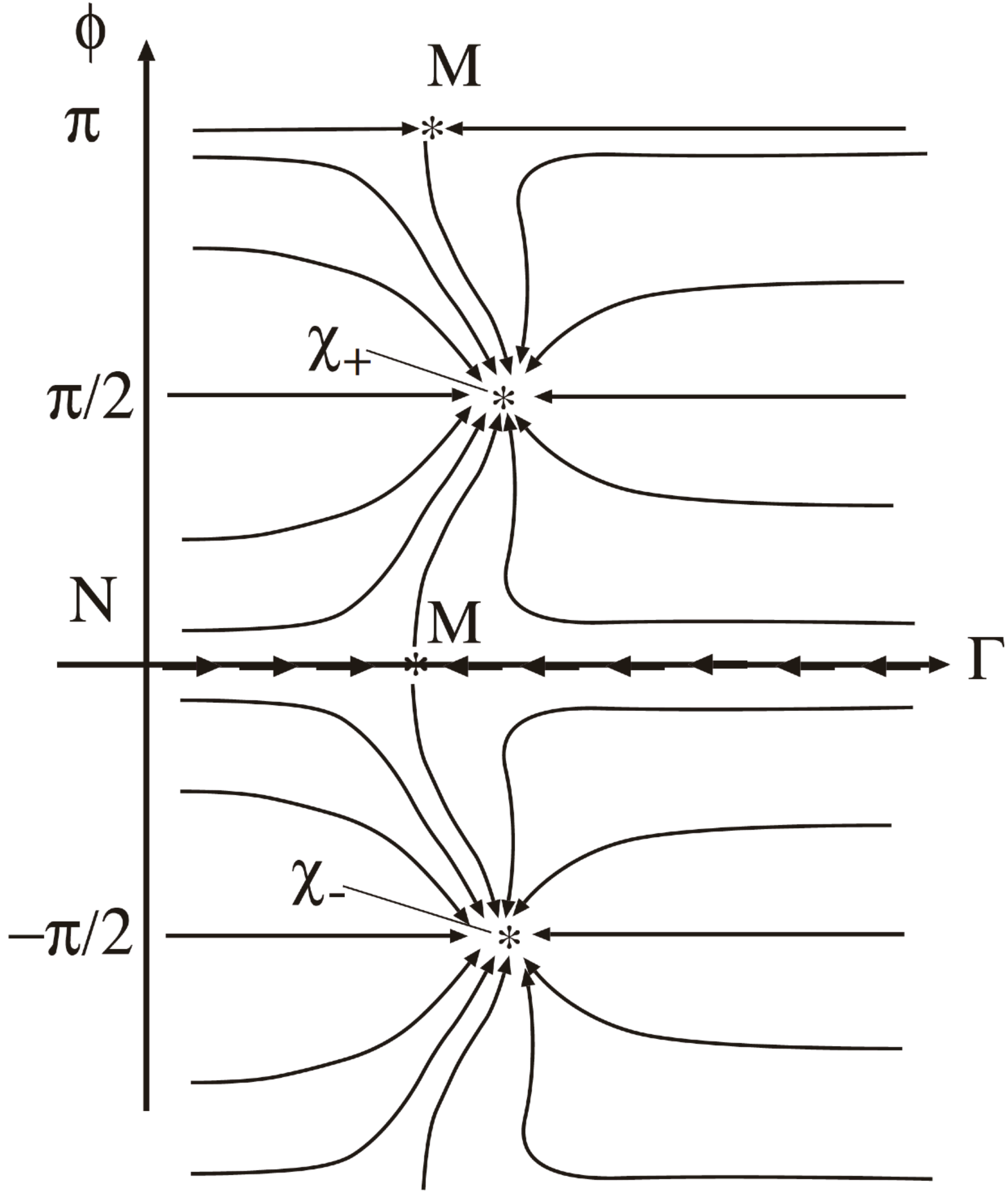
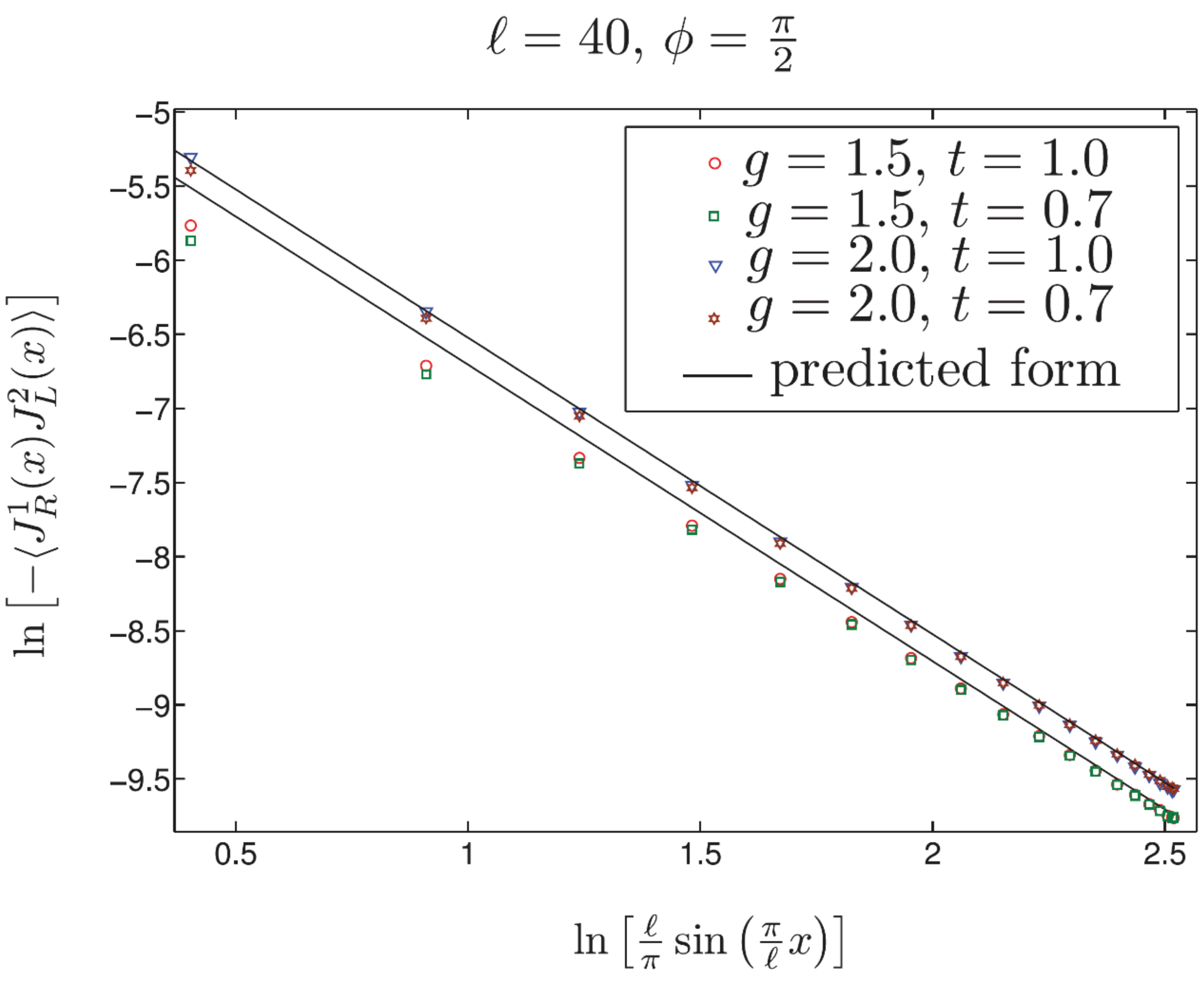


Conjectured
RG Flow Diagram
for 1 < K < 3

chiral χ±
exact solution
(chiral rotation of
Dirichlet/Neumann)

Rahmani, Hou et al. 2012



Chiral fixed point agree w/ BCFT prediction

M Fixed Point

$$G_{12}(g = 1.5) = G_{21}(g = 1.5) \approx -0.55\,\frac{e^2}{h},$$

$$G_{12}(g = 2.0) = G_{21}(g = 2.0) \approx -0.62\,\frac{e^2}{h},$$

$$G_{12}(g = 2.5) = G_{21}(g = 2.5) \approx -0.665\,\frac{e^2}{h}.$$

Rahmani, Hou et al. 2012

$$g = K$$

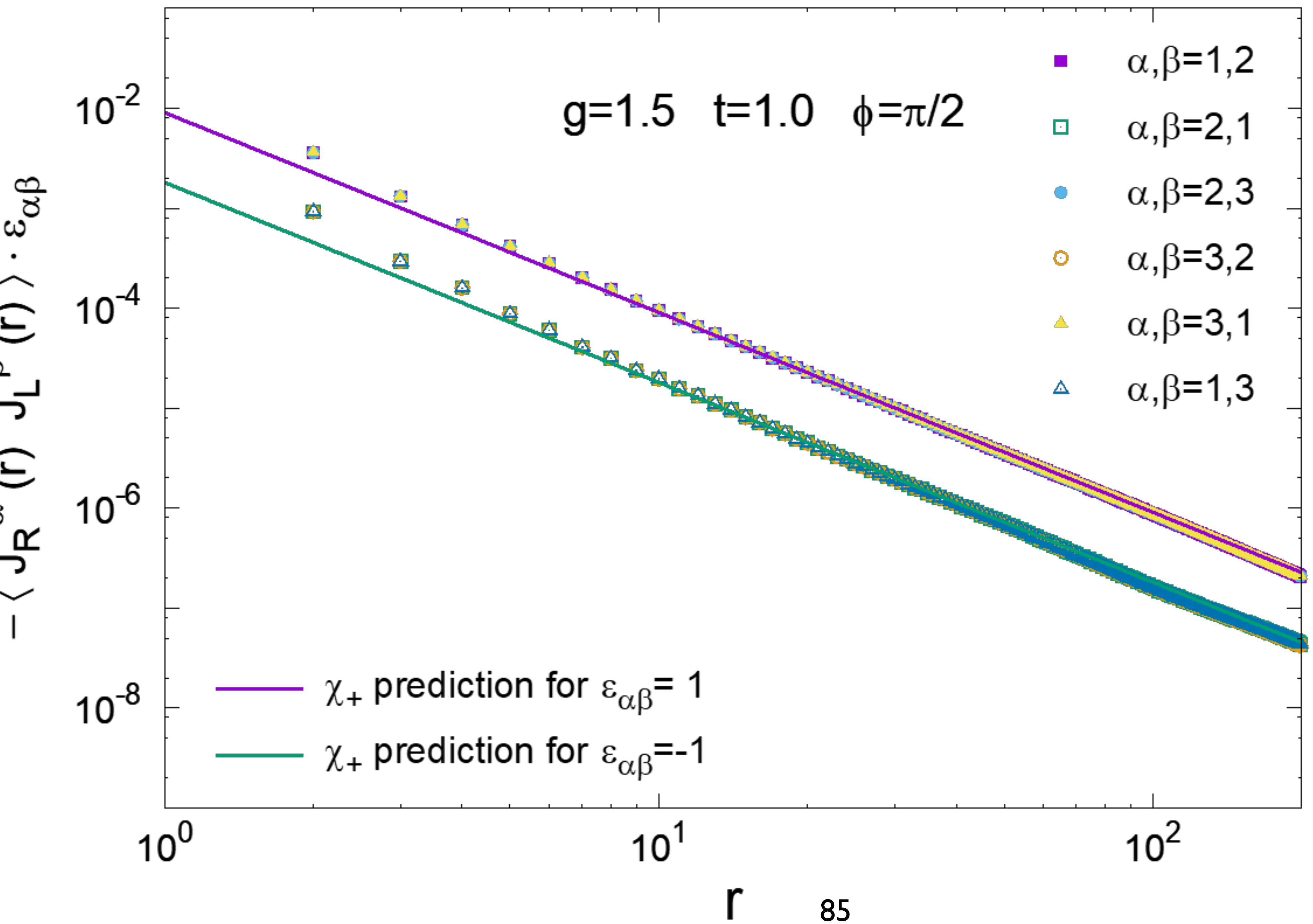
conjecture

$$G_{jj'} = \frac{2K\gamma}{2K + 3\gamma - 3K\gamma}\,\frac{e^2}{2\pi}(3\delta_{jj'} - 1),$$

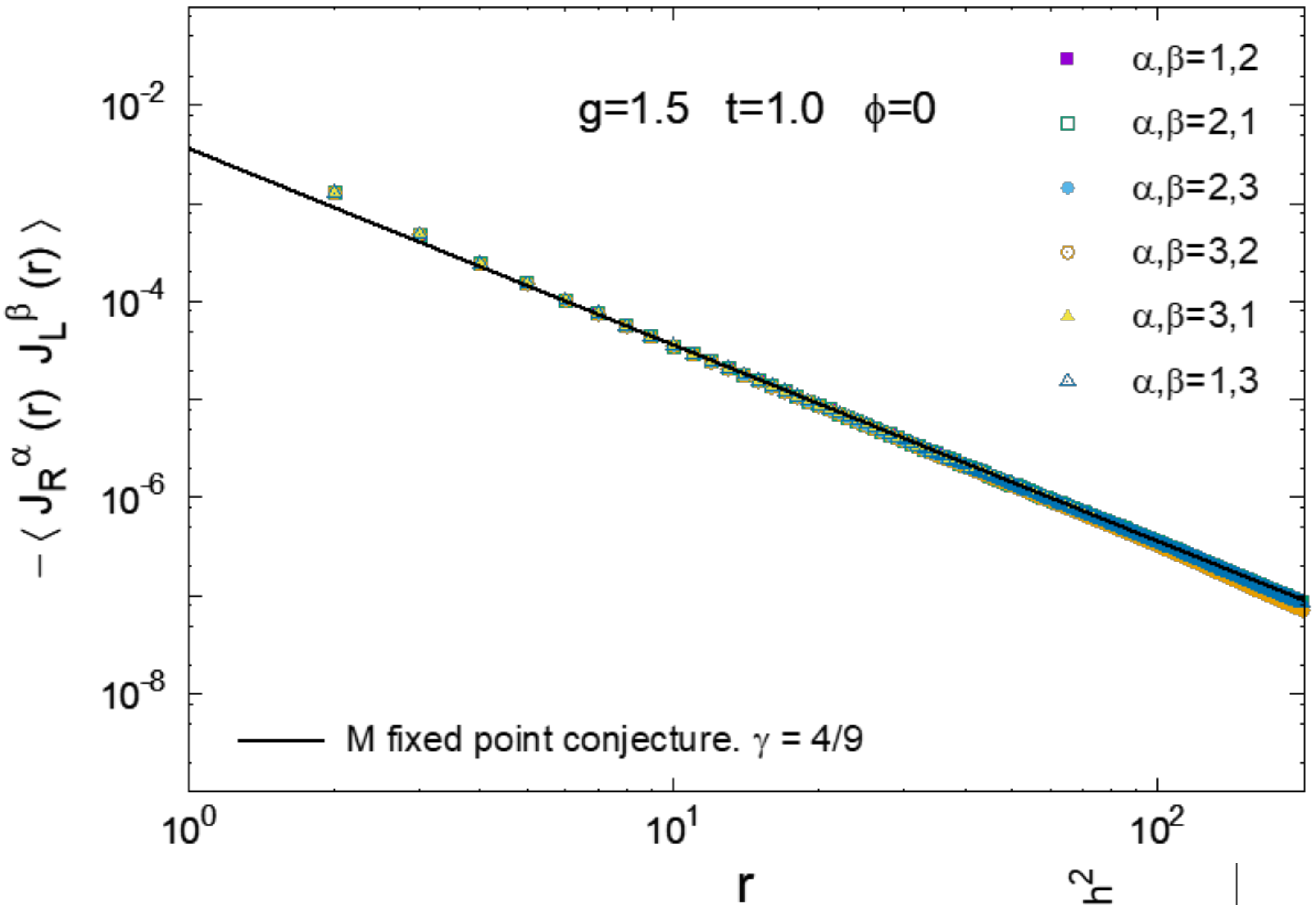
$$\gamma \sim \frac{4}{9}$$

Chiral F.P. in iDMRG+Window

C.-Y. Lo, Y.-J. Kao, P. Chen et al., preliminary unpublished



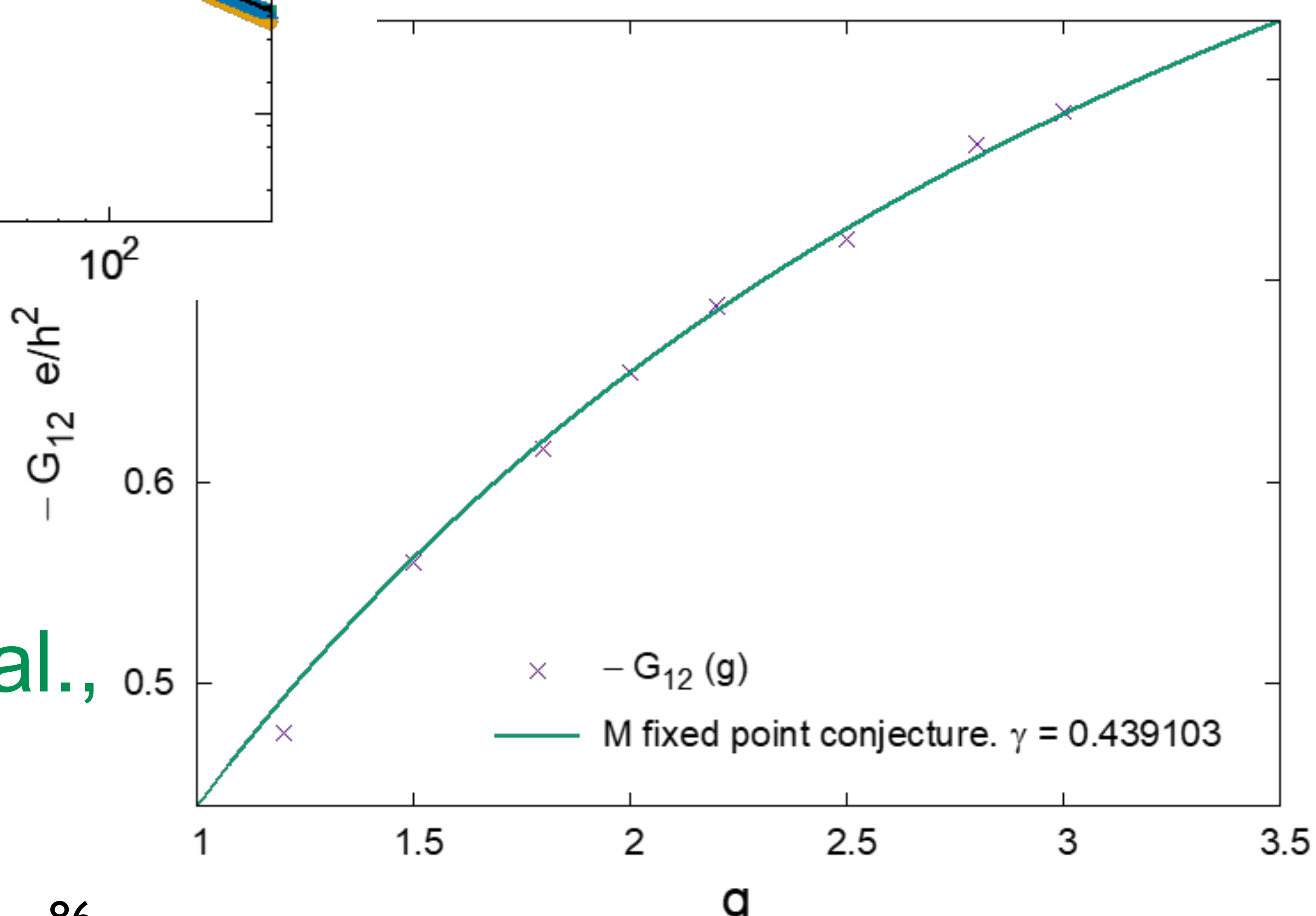
M Fixed Point in iDMRG+Window



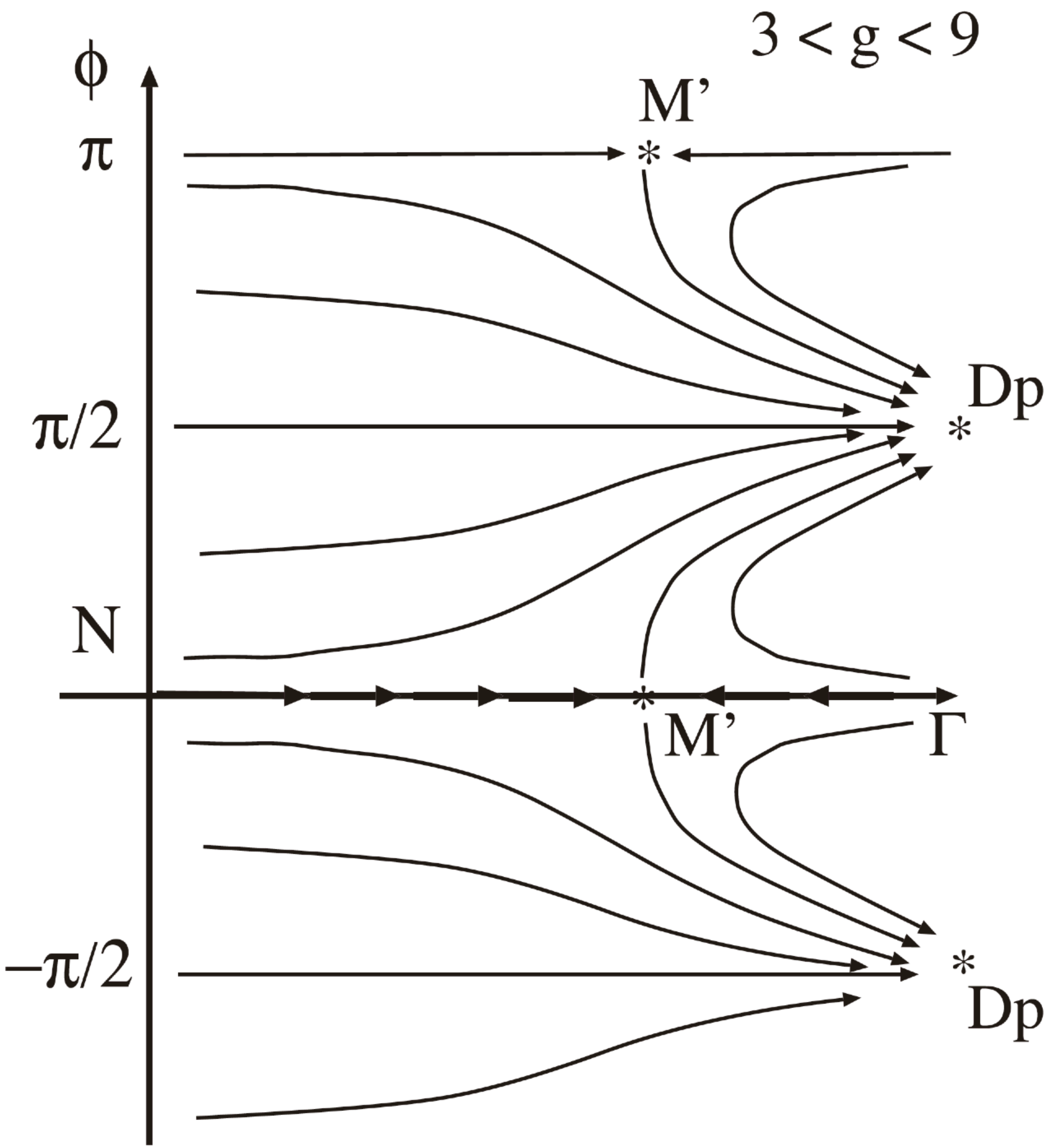
Rahmani et al. conjecture

$$G_{jj'} = \frac{2K\gamma}{2K + 3\gamma - 3K\gamma}\frac{e^2}{2\pi}(3\delta_{jj'} - 1),$$

$$\gamma \sim \frac{4}{9}$$

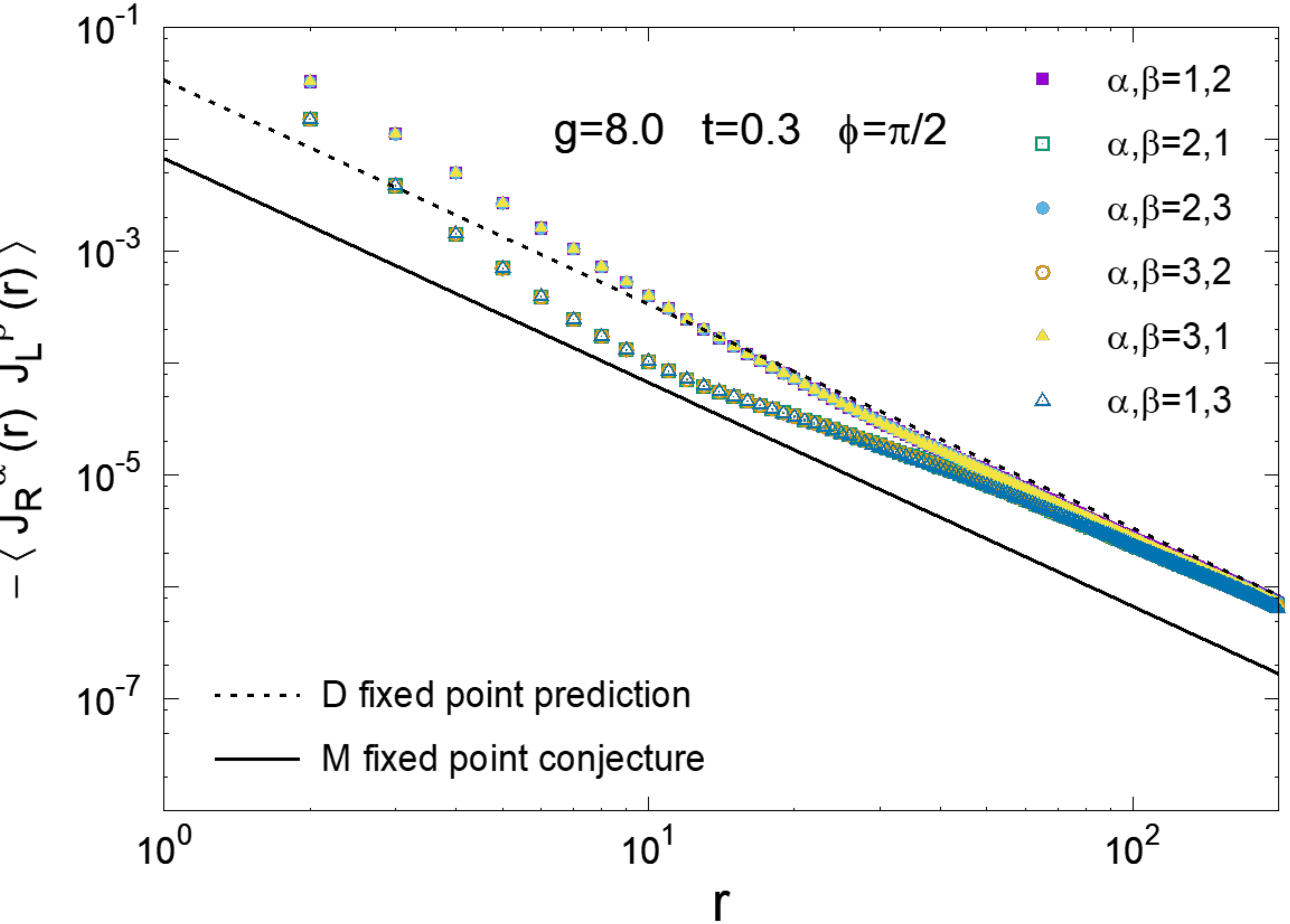


C.-Y. Lo, Y.-J. Kao, P. Chen et al., preliminary unpublished



3 < g < 9
ϕ
π
M’
π/2
Dp
N
M’
Γ
−π/2
Dp

Dp Fixed Point



C.-Y. Lo, Y.-J. Kao, P. Chen et al., preliminary unpublished

Conclusions (Part 2)

Classification of conformally invariant boundary conditions of irrational CFTs: open problem (generally unknown)

In particular, $c \geqq 2$ free boson CFTs (multicomponent TLLs) have various nontrivial conformal boundary conditions with many applications

A few known exact solutions, and many more presumed to exist from RG arguments

Refined numerical methods can help verifying predictions and discovering more