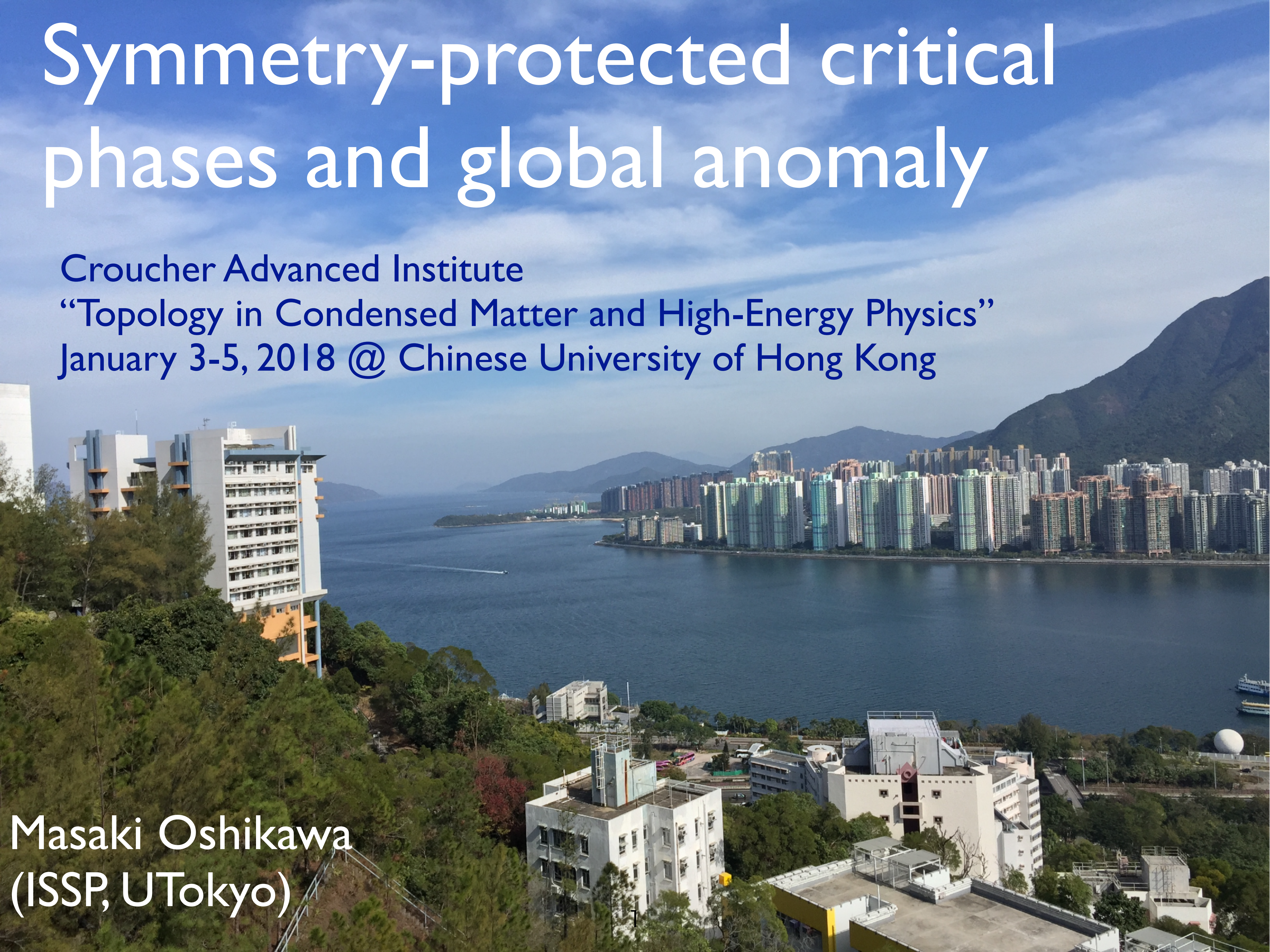


Symmetry-protected critical phases and global anomaly

Croucher Advanced Institute
"Topology in Condensed Matter and High-Energy Physics"
January 3-5, 2018 @ Chinese University of Hong Kong

Masaki Oshikawa
(ISSP, UTokyo)

Lecture I:
Anomaly and Condensed Matter Physics

Lecture II:
Symmetry-Protected Critical Phases and Global Anomaly

Anomaly (physics)

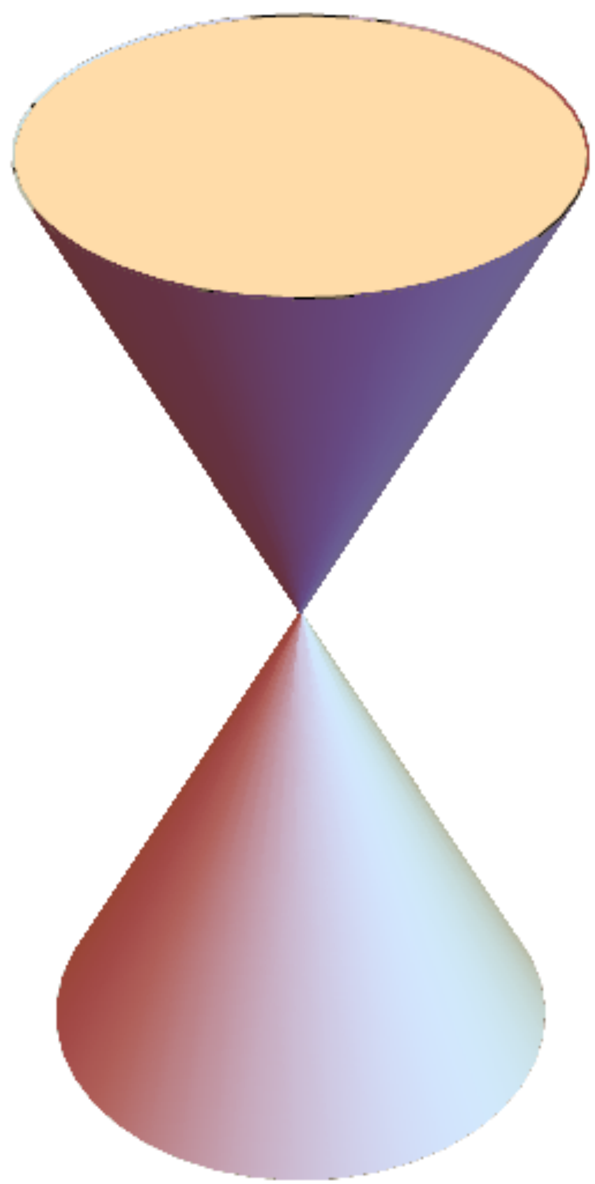
From Wikipedia, the free encyclopedia

In quantum physics an **anomaly** or **quantum anomaly** is the failure of a symmetry of a theory's classical action to be a symmetry of any regularization of the full quantum theory.[1][2] In classical

Dirac Fermion

$$\mathcal{L} = \bar{\psi}\left(i\hbar\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$$



$$\omega = \pm\sqrt{\vec{p}^2 + m^2}$$

massless
(m=0)

$$\omega = \pm|\vec{p}|$$

"Dirac cone"

Axial Symmetry and Current

Massless Dirac fermion Lagrangian density

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

"vector" U(1) symmetry ⇒ charge current conservation

$$\psi \to e^{i\theta_V} \psi \qquad \partial_\mu j^\mu = 0$$

$$\bar{\psi} \to e^{-i\theta_V} \bar{\psi} \qquad j^\mu = \bar{\psi}\gamma^\mu\psi$$

$$\{\gamma^\mu, \gamma^5\} = 0 \qquad \left(\gamma^5\right)^2 = 1$$ in **even** space-time dimensions

"axial" U(1) symmetry if *m*=0 ⇒ axial current conservation

$$\psi \to e^{i\theta_A} \psi \qquad \partial^\mu j^5_\mu = 0$$

$$\bar{\psi} \to e^{i\theta_A} \bar{\psi} \qquad j^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi$$

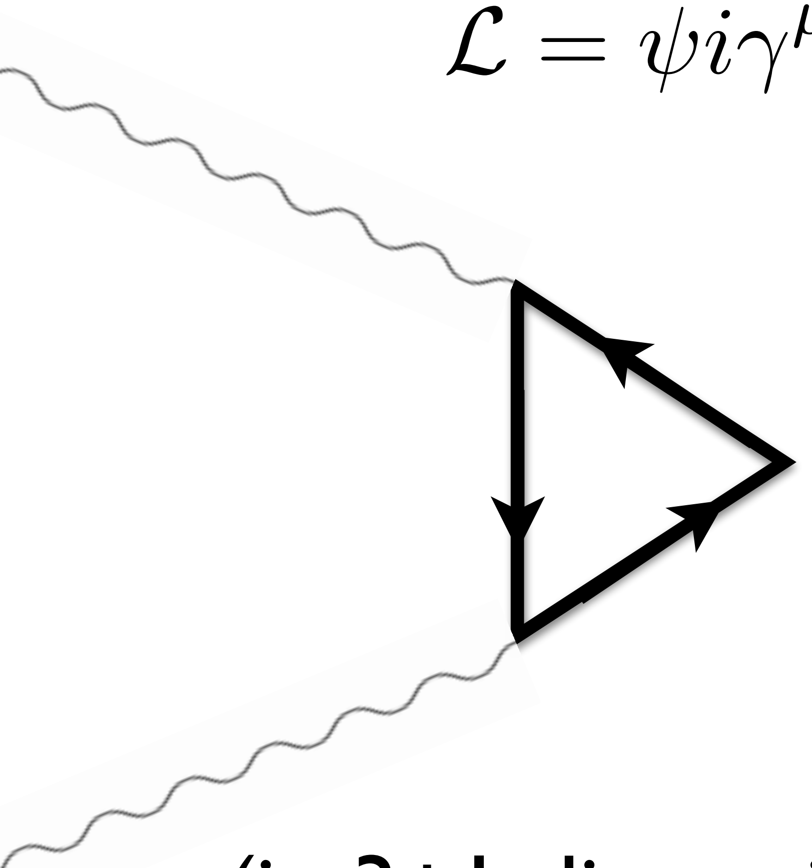
U(1) Chiral Anomaly

Noether's theorem ("classical"):
Massless Dirac fermion ⇒ two conserved currents

However, one of these conservation laws is inevitably broken in quantum theory through "regularization" of UV divergence

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu - i A_\mu) \psi$$

Adler/Bell-Jackiw (1969)



$\gamma^\mu \gamma^5$

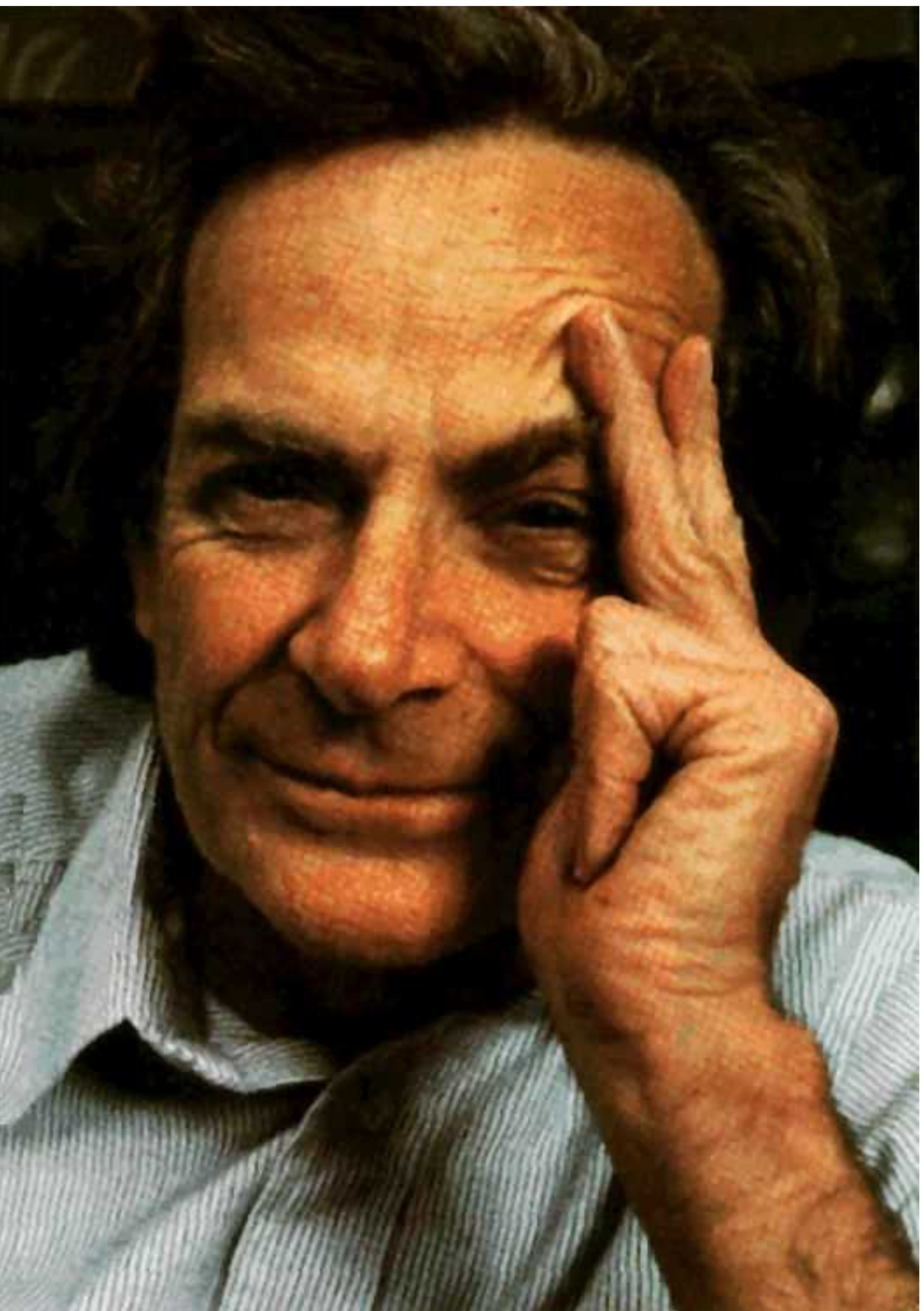
(in 3+1 dimensions)

Anomalous non-conservation of axial current!

$$\partial^\mu j^5_\mu = \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

Decay of neutral pion $\pi^0 \to \gamma\gamma$

Regularization/Renormalization



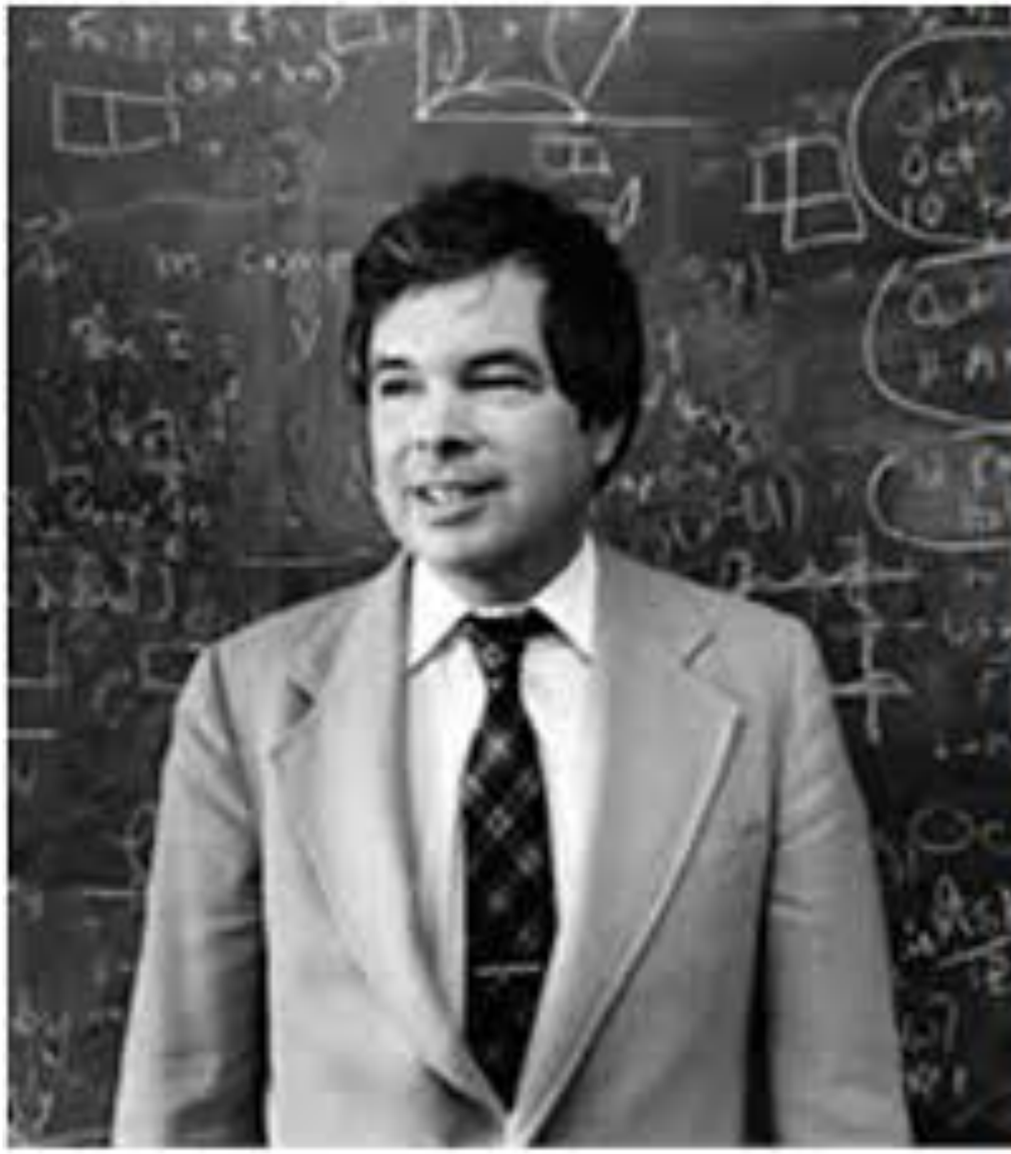
"renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics
under the rug."
Richard Feynman, in
Nobel Lecture (1965)

Modern Understanding of Renormalization

Field theory =
universal long-distance behavior
of lattice model /
condensed matter systems

Exact symmetry in the lattice model remains exact in the long-distance limit
→ no anomaly?

How can we understand anomaly
in this context?



Kenneth G. Wilson
(1936-2013)

Chiral Anomaly in 1+1 Dim.

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)\psi \qquad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

ψ_R, ψ_L Right mover, Left mover

U(1) x U(1) symmetry $\psi_{R,L} \to \psi_{R,L} e^{i\theta_{R,L}}$

$n_{R,L} \equiv \psi^{\dagger}_{R,L}\psi_{R,L}$ conserved individually?

However, one of the conservation law is broken

$$\partial^{\mu} j^{5}_{\mu} = \frac{1}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \qquad \frac{\partial}{\partial t}(n_R - n_L) \propto E$$

Chiral Anomaly

Chiral Symmetry on Lattice

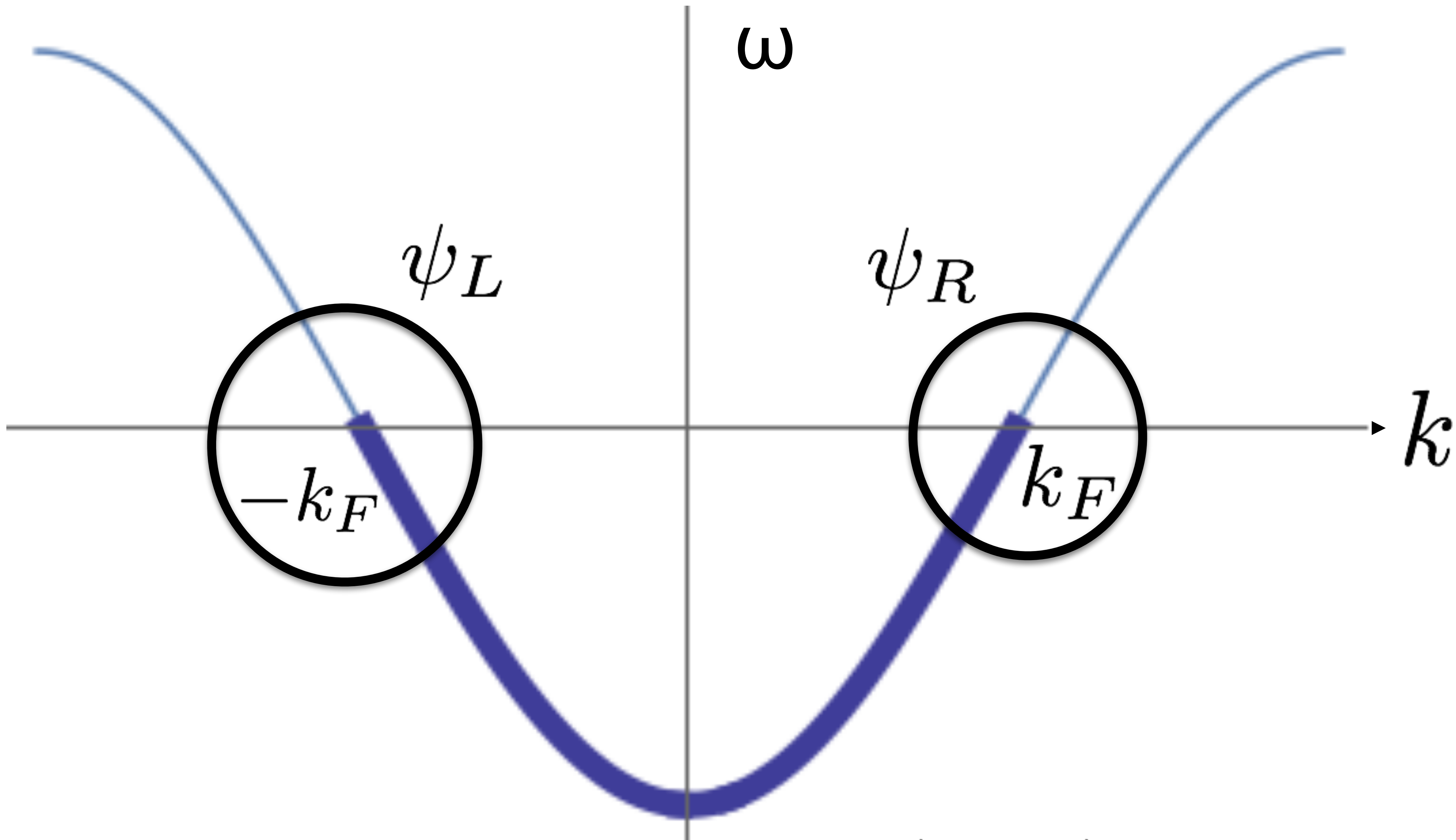
If we can realize Dirac fermion on lattice with
exact chiral symmetry, the chiral symmetry should persist
$\Rightarrow$ contradiction with chiral anomaly

In particular, if we can realize the right-moving and left-moving "Weyl fermion" individually on the lattice, the exact chiral symmetry would follow

This suggests that, we cannot realize chiral symmetry
exactly in a lattice model,
and that we cannot realize an individual right-moving/left-moving "Weyl fermion" in a lattice

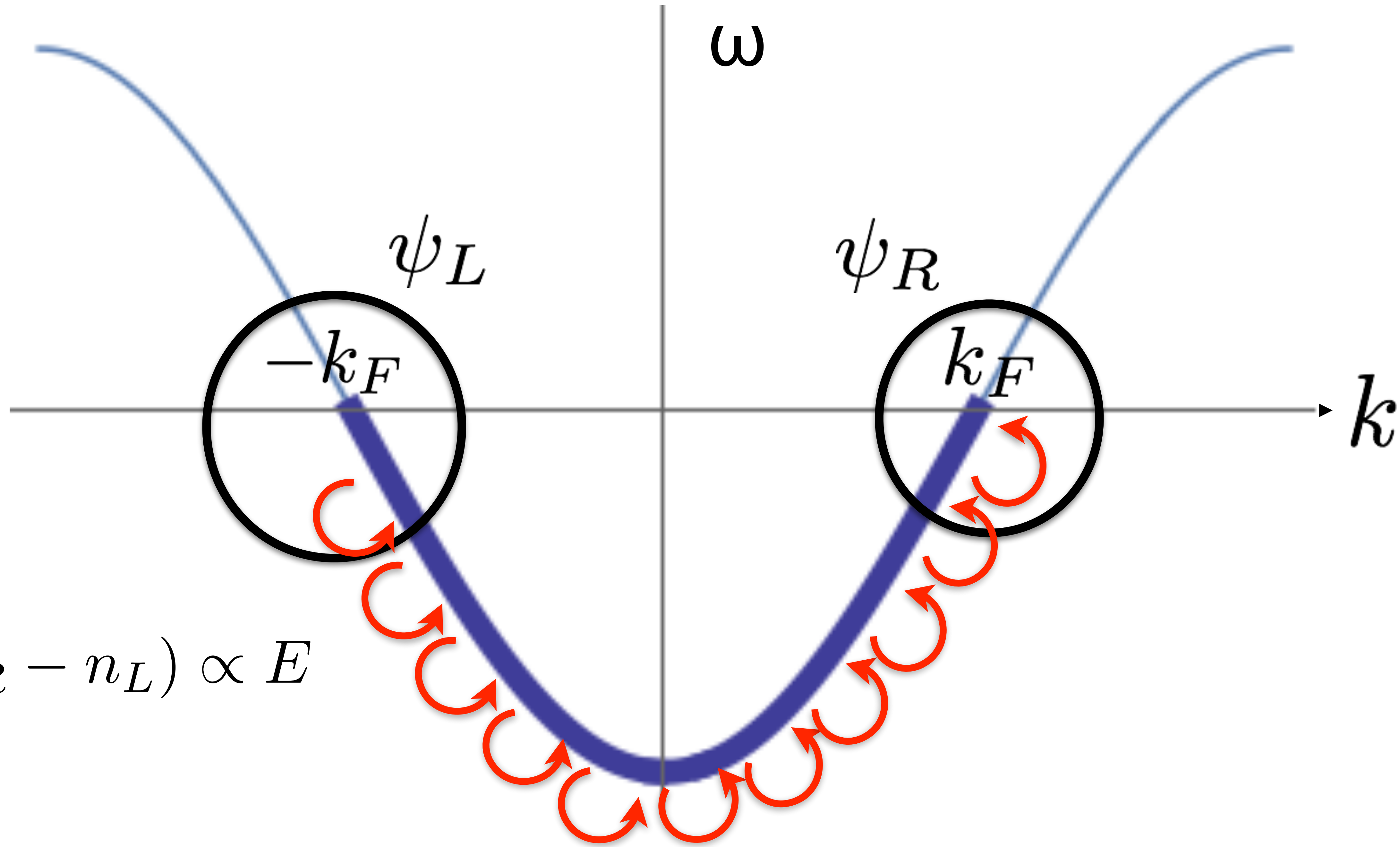
$\Leftrightarrow$ **Nielsen-Ninomiya theorem**

Nielsen-Ninomiya in 1+1D



Periodicity of the momentum space (Brillouin zone) → ψ_R, ψ_L always appear in pair!

Chiral Anomaly in 1+1D



$$\frac{\partial}{\partial t}(n_R - n_L) \propto E$$

Acceleration of electrons by electric field!
universal in low-energy limit (topological quantization)

Nuclear Physics B193 (1981) 173–194

ABSENCE OF NEUTRINOS ON A LATTICE
(II). Intuitive topological proof

H.B. NIELSEN

The Niels Bohr Institute, University of Copenhagen, and NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

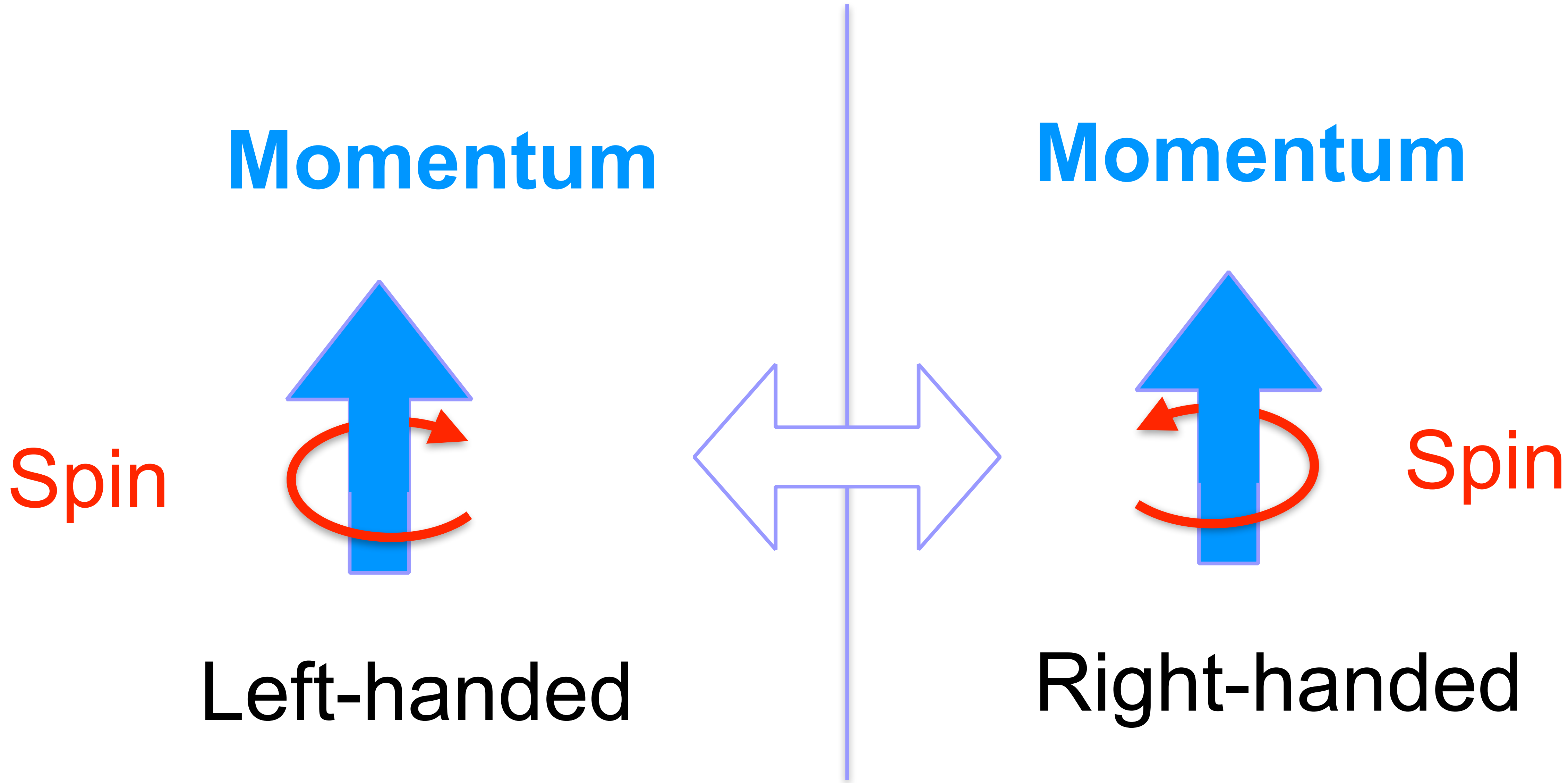
M. NINOMIYA

Rutherford Laboratory, Chilton, Didcot, OXON OX11 0QX, England

Same expectation (absence of single Weyl fermion on lattice) for 3+1D based on chiral anomaly

Proof is a bit more complicated

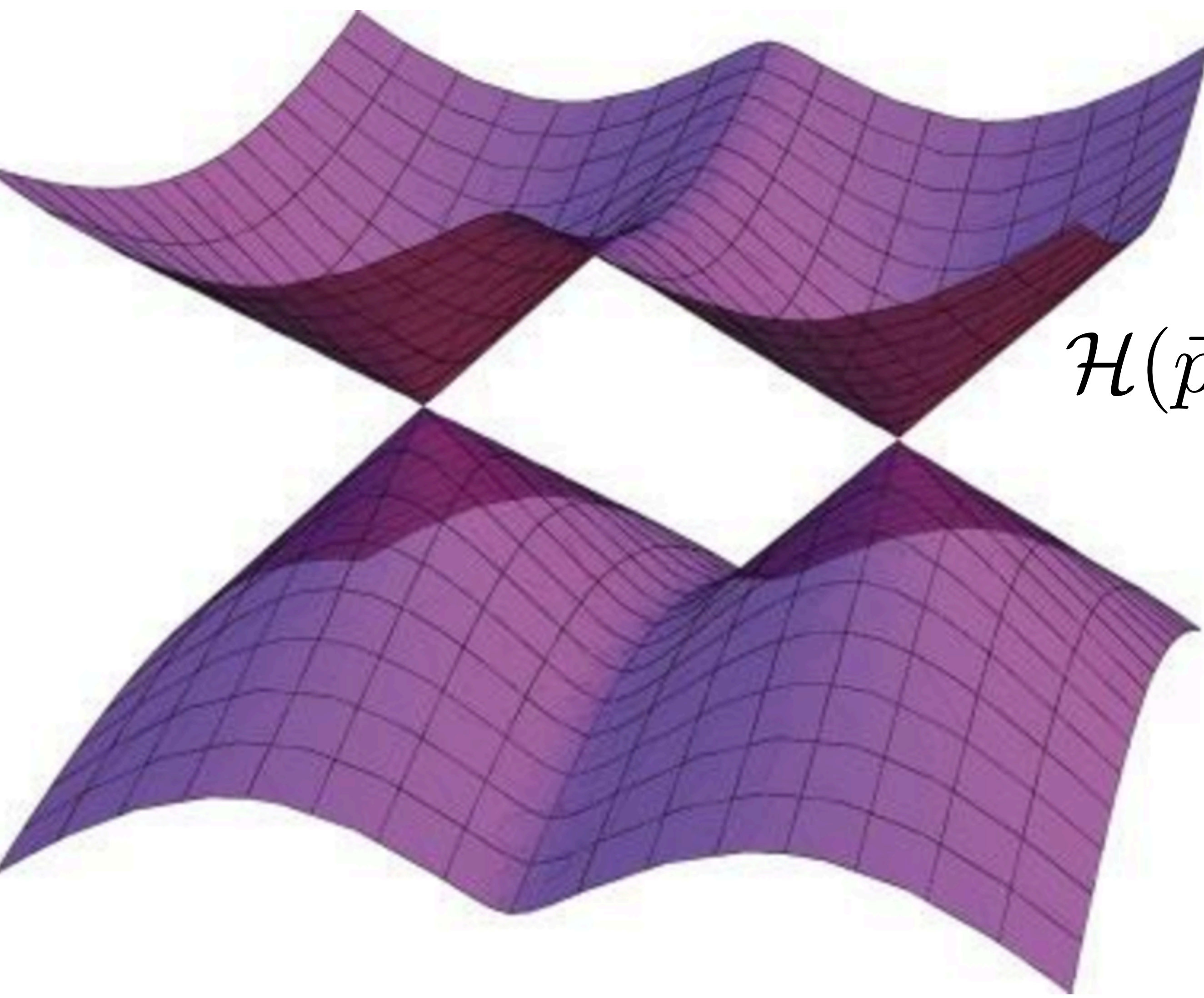
Chirality in 3+1 D



(Helicity = Chirality if massless)

Weyl Fermion in 3+1D & Lattice

$$\mathcal{H}(\vec{p})|\Psi_\alpha(\vec{p})\rangle = \epsilon_\alpha(\vec{p})|\Psi_\alpha(\vec{p})\rangle$$ band structure



Touching of two bands
= Weyl point

$$\mathcal{H}(\vec{p}) \sim \epsilon(\vec{p}^*) + \sum_{\mu,\nu} V_{\mu\nu}\sigma^\nu(p^\mu - p^{*\mu})$$

chirality = $\operatorname{sgn}\det V$

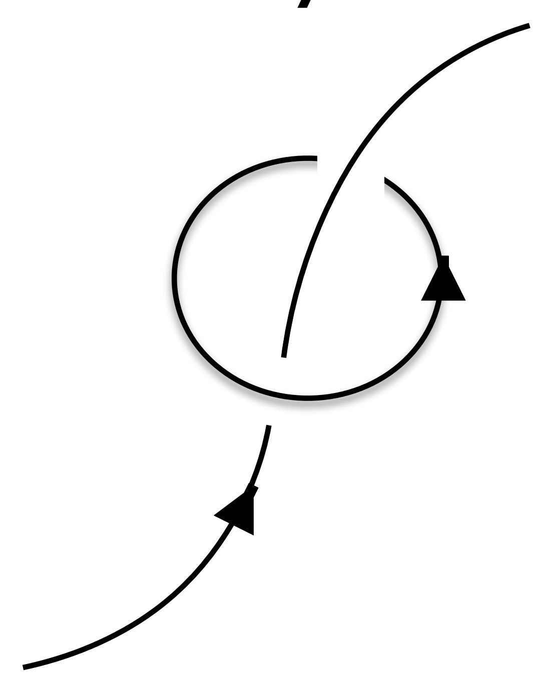
Vortex Lines

$$\langle a | \Psi_\alpha(\vec{p}) \rangle = 0$$

2 conditions (real part = imaginary part = 0)
3 parameters
$\rightarrow$ solution consists of curves in the momentum space

"vortex line"

The direction of the vortex line is defined by
"vorticity" i.e.
the winding of the complex phase
around the vortex line

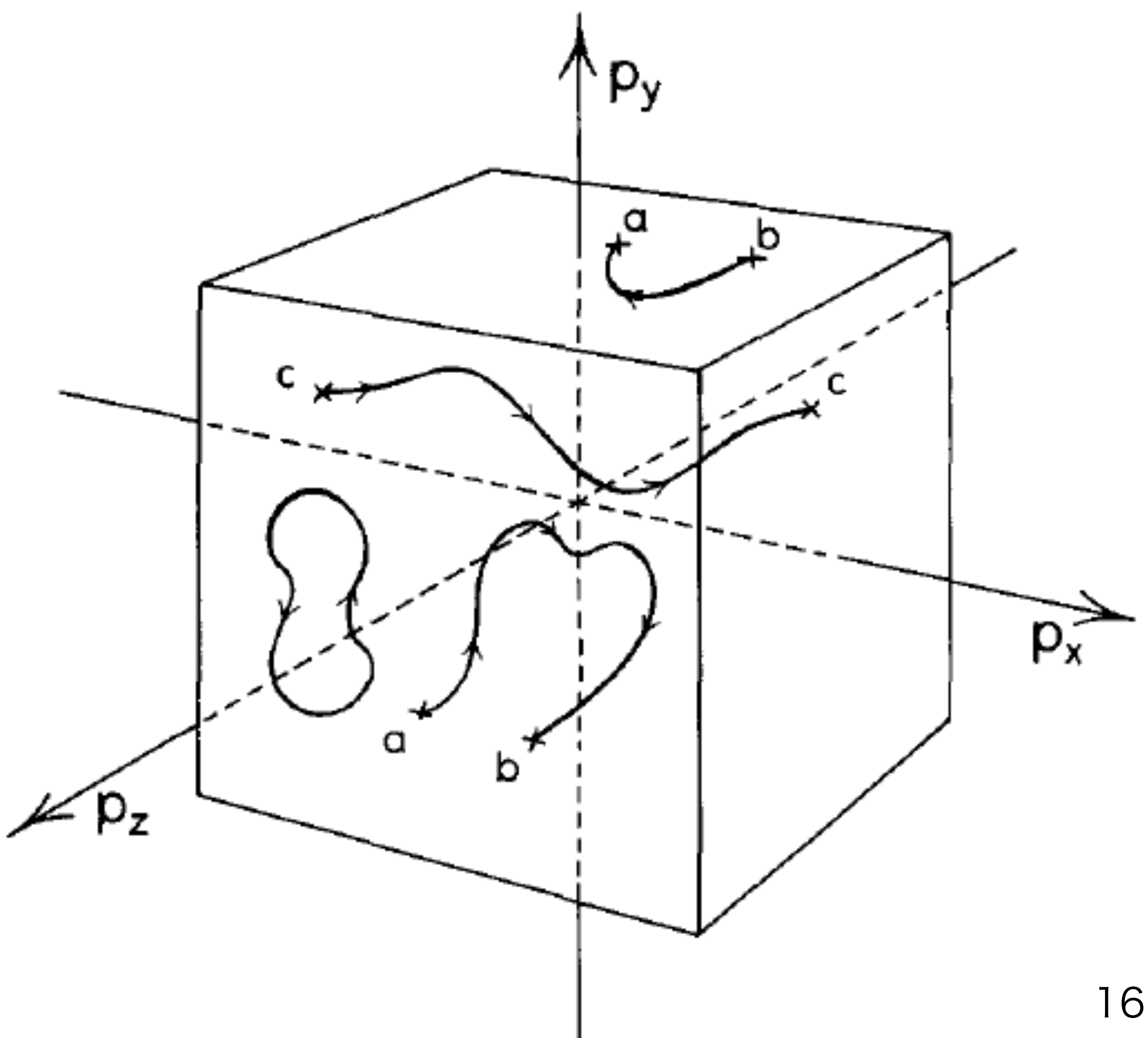


Weyl (band-touching) Points

Two bands degenerate at the Weyl points

$\Rightarrow$ there is always a solution for $\langle a|\Psi(\vec{p})\rangle = 0$

by considering a linear superposition of two states



Weyl point:
source/sink of "vortex line"
in 3D momentum space

Each vortex line should have
an "origin" and "endpoint"
$\Rightarrow$ Weyl points always appear
in pair of opposite chiralities

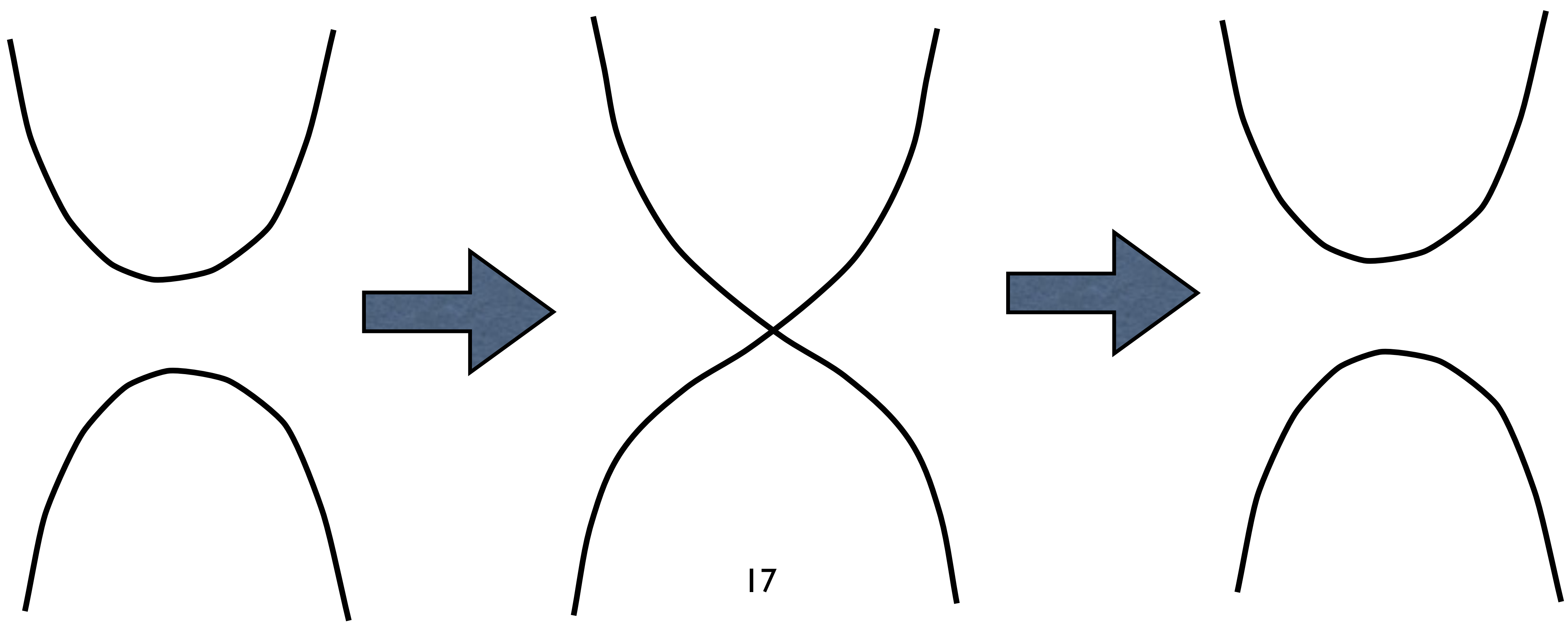
Dirac Fermions in 2+1D

$$\gamma^0 = \sigma^x, \gamma^1 = i\sigma^y, \gamma^2 = i\sigma^z$$

$$\epsilon(\vec{p}) \sim p_x\sigma^x + p_y\sigma^y + m\sigma^z$$

Generic "band-touching" situation in CM

$m < 0$ $m=0$ $m>0$



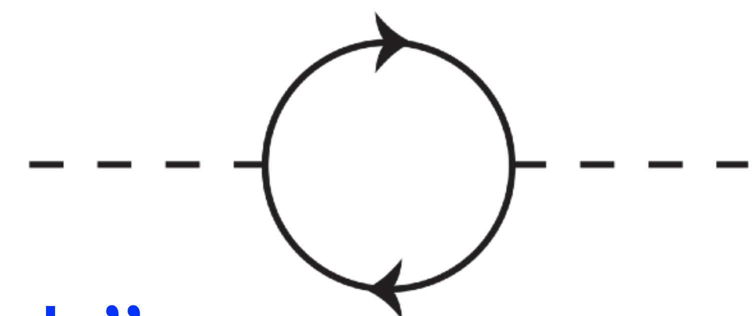
Parity anomaly in 2+1D

Field-theory calculation gives

Dirac mass

Pauli-Villars mass

$$\sigma_{xy} = \frac{e^2}{2h}\left(\mathrm{sgn} m + \mathrm{sgn} M\right)$$



"parity anomaly"

Massless Dirac fermion somehow has non-zero Hall conductivity (breaking the time-reversal symmetry "spontaneously")

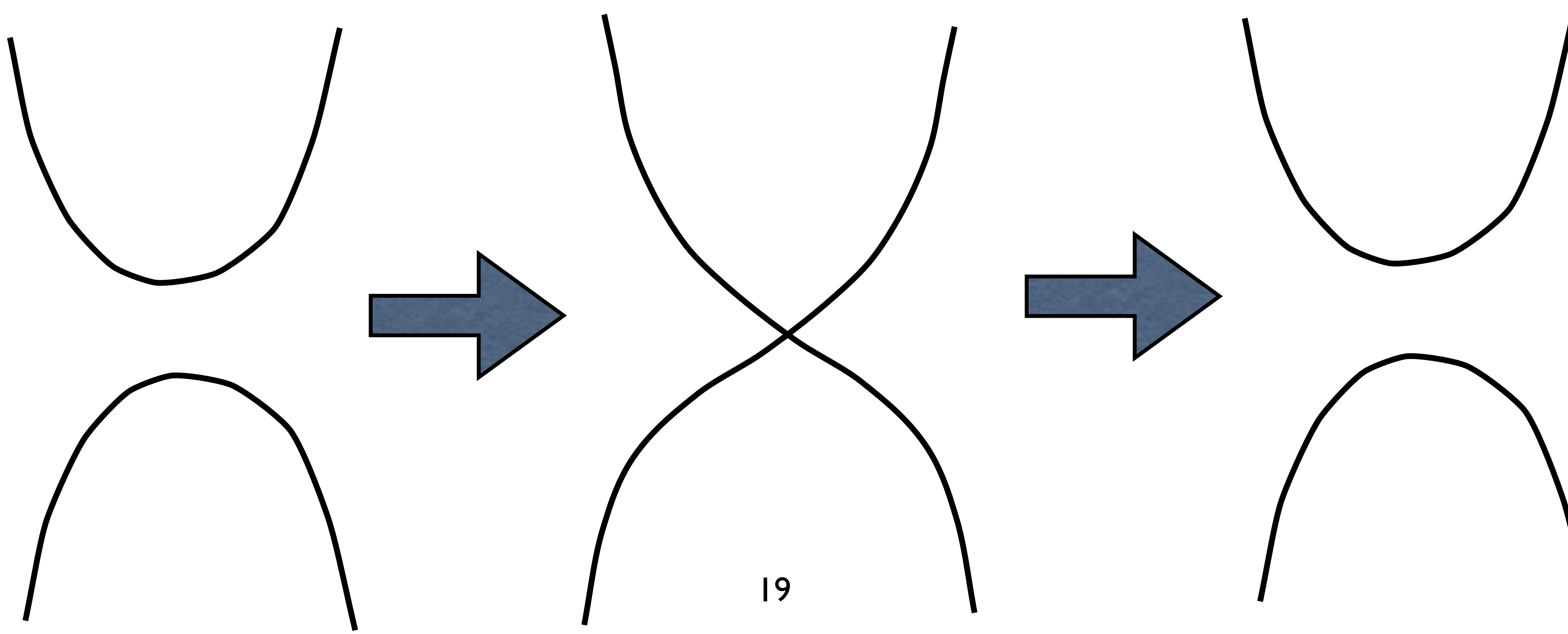
This implies that one cannot realize a single massless Dirac fermion in **time-reversal invariant** 2+1 dimensional lattice model

(Dirac fermions always appear in pairs)

— distinct from, but similar to Nielsen-Ninomiya theorem in even space-time dimensions

“Dimensional Reduction”

$m < 0$ $m=0$ $m>0$



2-dimensional momentum space
+ 1 external parameter

3-dimensional
momentum space

2+1 D massless Dirac fermion
at the critical point

3+1D Weyl fermion

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

VOLUME 51, NUMBER 24 PHYSICAL REVIEW LETTERS 12 DECEMBER 1983

Holonomy, the Quantum Adiabatic Theorem, and Berry's Phase

Barry Simon
Departments of Mathematics and Physics, California Institute of Technology, Pasadena, California 91125
(Received 18 October 1983)

Topological Invariant and the Quantization of the Hall Conductance

MAHITO KOHMOTO*

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\int_{\mathrm{MBZ}} d^2\vec{k}\,\nabla_{\vec{k}} \times \mathcal{A}(\vec{k})$$

$$= \frac{e^2}{h}\text{“Chern Number”}$$

“total vorticity”

Dimensional Reduction & Chern Number

PHYSICAL REVIEW B VOLUME 50, NUMBER 23 15 DECEMBER 1994-I

Quantized Hall conductivity of Bloch electrons: Topology and the Dirac fermion

Masaki Oshikawa*
Institute for Solid State Physics, University of Tokyo, Roppongi, Minato-ku Tokyo 106, Japan
(Received 28 June 1994)

Selected for a Viewpoint in *Physics*

PHYSICAL REVIEW B **83**, 205101 (2011)

Topological semimetal and Fermi-arc surface states in the electronic structure of pyrochlore iridates

Xiangang Wan,[1] Ari M. Turner,[2] Ashvin Vishwanath,[2,3] and Sergey Y. Savrasov[1,4]
[1]*National Laboratory of Solid State Microstructures and Department of Physics, Nanjing University, Nanjing 210093, China*
[2]*Department of Physics, University of California, Berkeley, California 94720, USA*
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(Received 23 February 2011; published 2 May 2011)

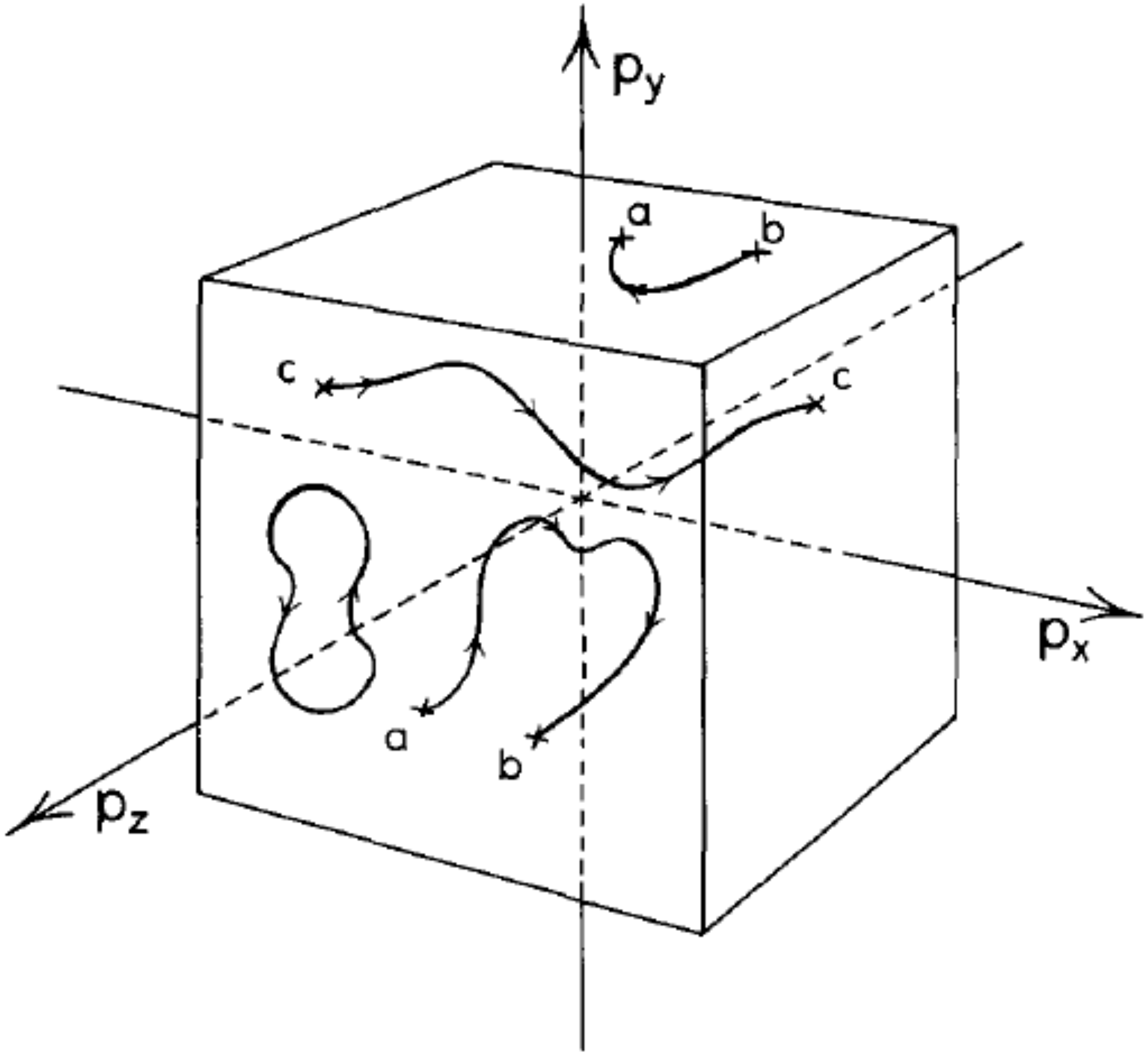


Figure: Nielsen-Ninomiya (1981)

Chern number (2+1D)
= total vorticity appearing in a 2D cross section of 3D momentum space

$$\sigma_{xy} = \frac{e^2}{2h}(\mathrm{sgn}m + \mathrm{sgn}M)$$

$$\Delta\text{Chern Number} = \frac{1}{2}\sum_j \Delta\mathrm{sgn}m_j$$

M.O. 1994

$$\sigma_{xy} = \frac{e^2}{h}\text{“Chern Number”}$$

“Corollary”

If a single massless Dirac fermion is realized in
a time-reversal invariant lattice model in 2+1D,

$$\sigma_{xy} = 0$$

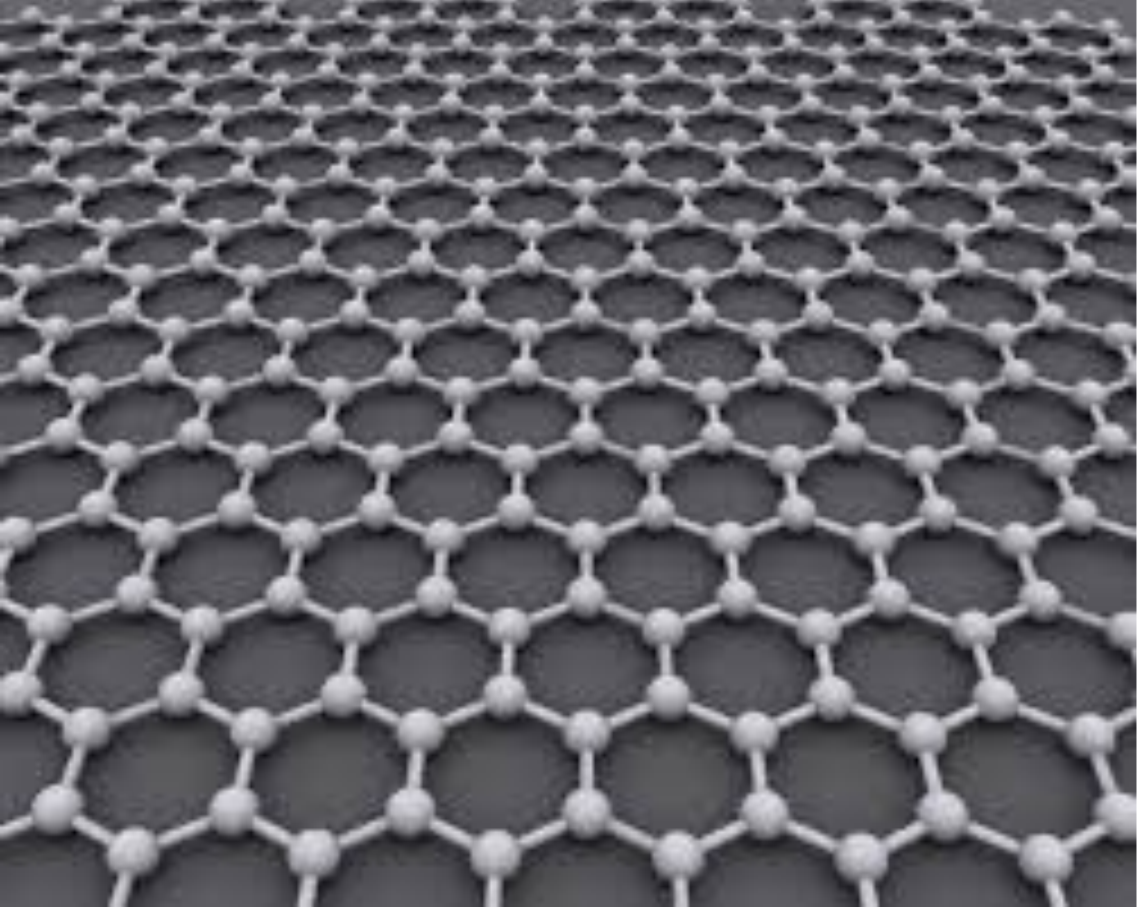
by symmetry

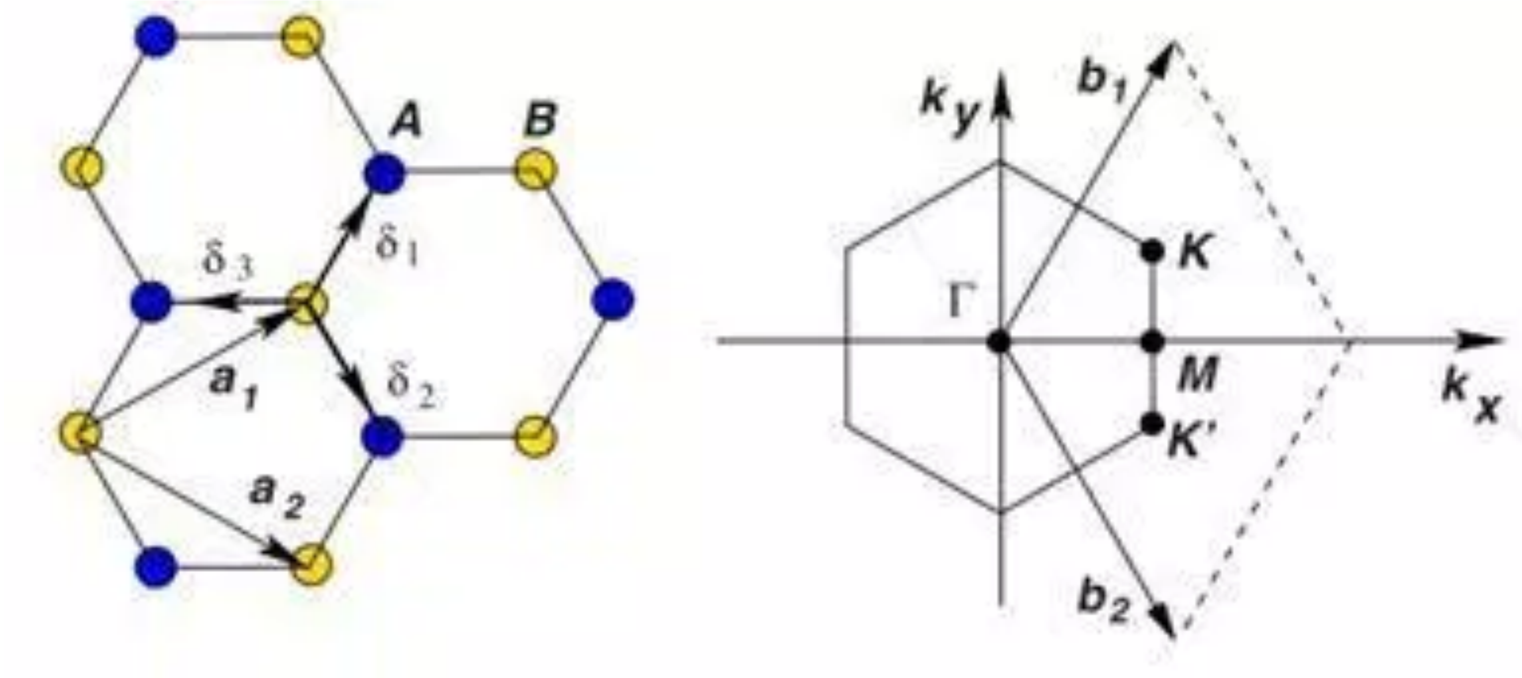
Now, with a perturbation which opens a mass gap,

$$\sigma_{xy} = \frac{1}{2}\frac{e^2}{h}\mathrm{sgn} m$$

which would contradict the TKNN quantization

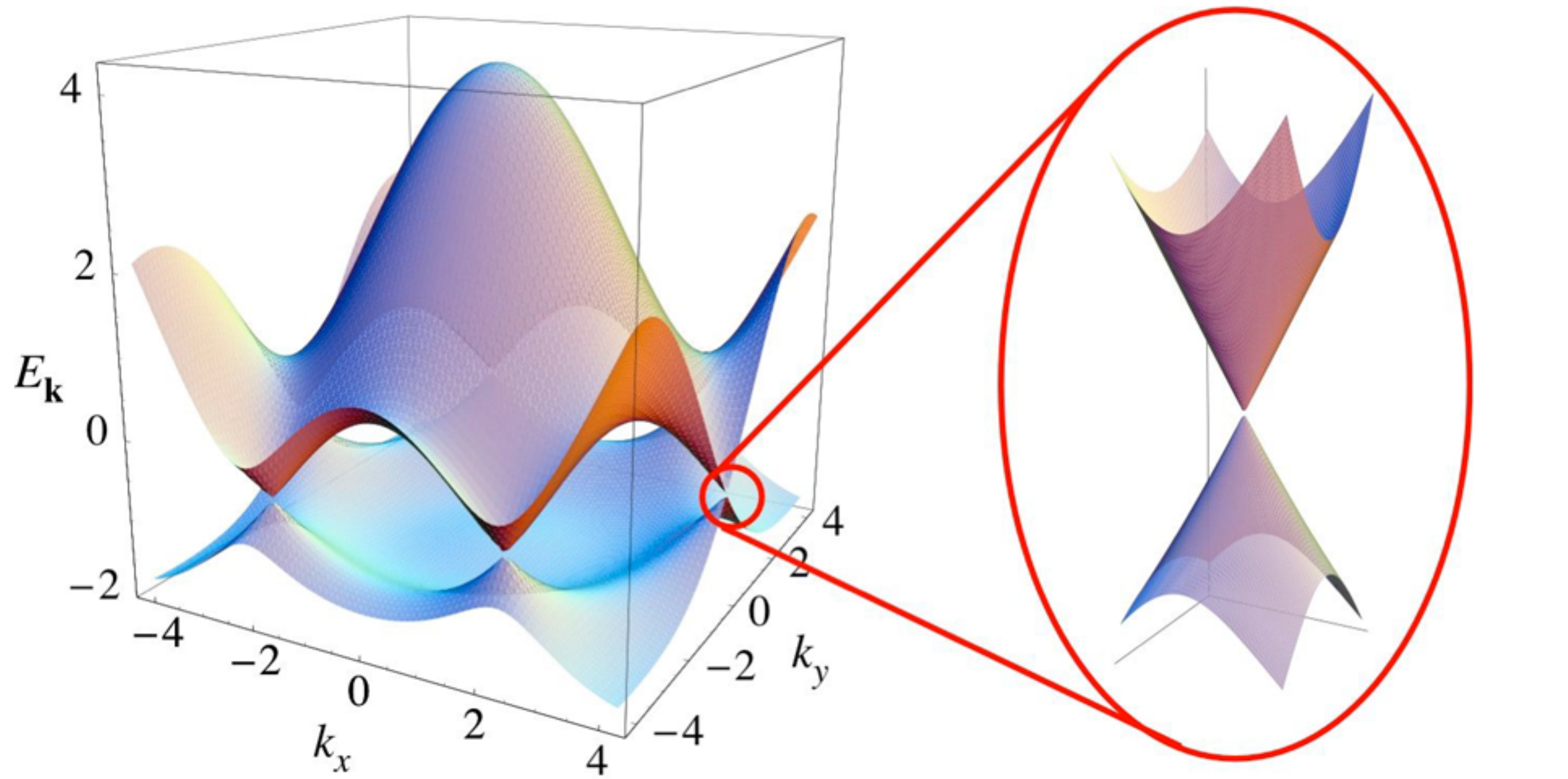
By contradiction, a single massless Dirac fermion in 2+1D
CANNOT be realized in a TR-invariant lattice model in 2+1D





Graphene

time-reversal invariant,
two Dirac points at K and K'



Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane
Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.

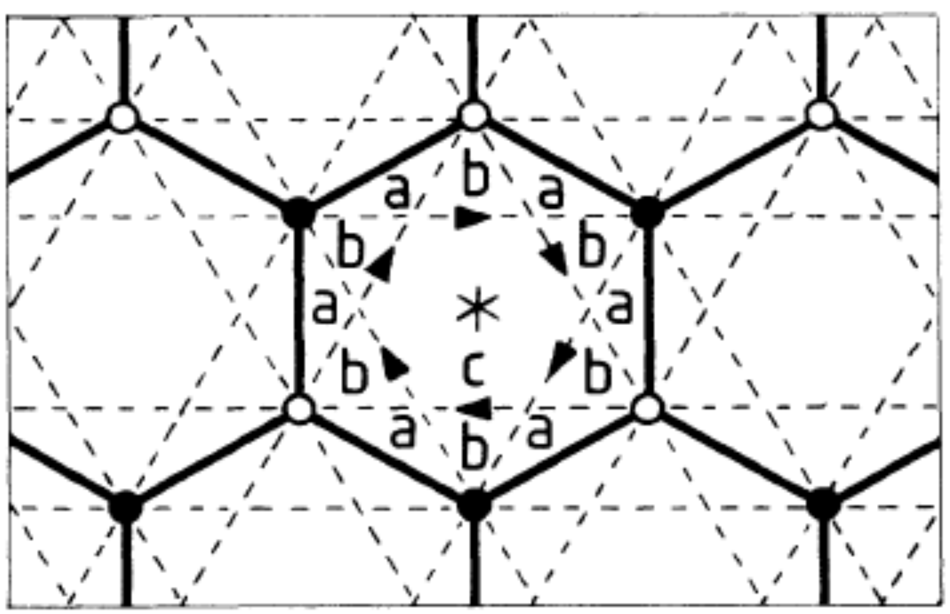


FIG. 1. The honeycomb-net model ("2D graphite") showing nearest-neighbor bonds (solid lines) and second-neighbor bonds (dashed lines). Open and solid points, respectively, mark the A and B sublattice sites. The Wigner-Seitz unit cell is conveniently centered on the point of sixfold rotation symmetry

Anomaly & "No-Go" Theorems

Anomaly in quantum field theory implies
"no-go" theorem for lattice model

Chiral anomaly → Absence of chiral Dirac fermion on lattice
(Nielsen-Ninomiya theorem for
even space-time dimensions)

Parity anomaly → Absence of single massless Dirac fermion on
time-reversal invariant lattice model in 2+1D

Any "loophole" to realize them on lattice?

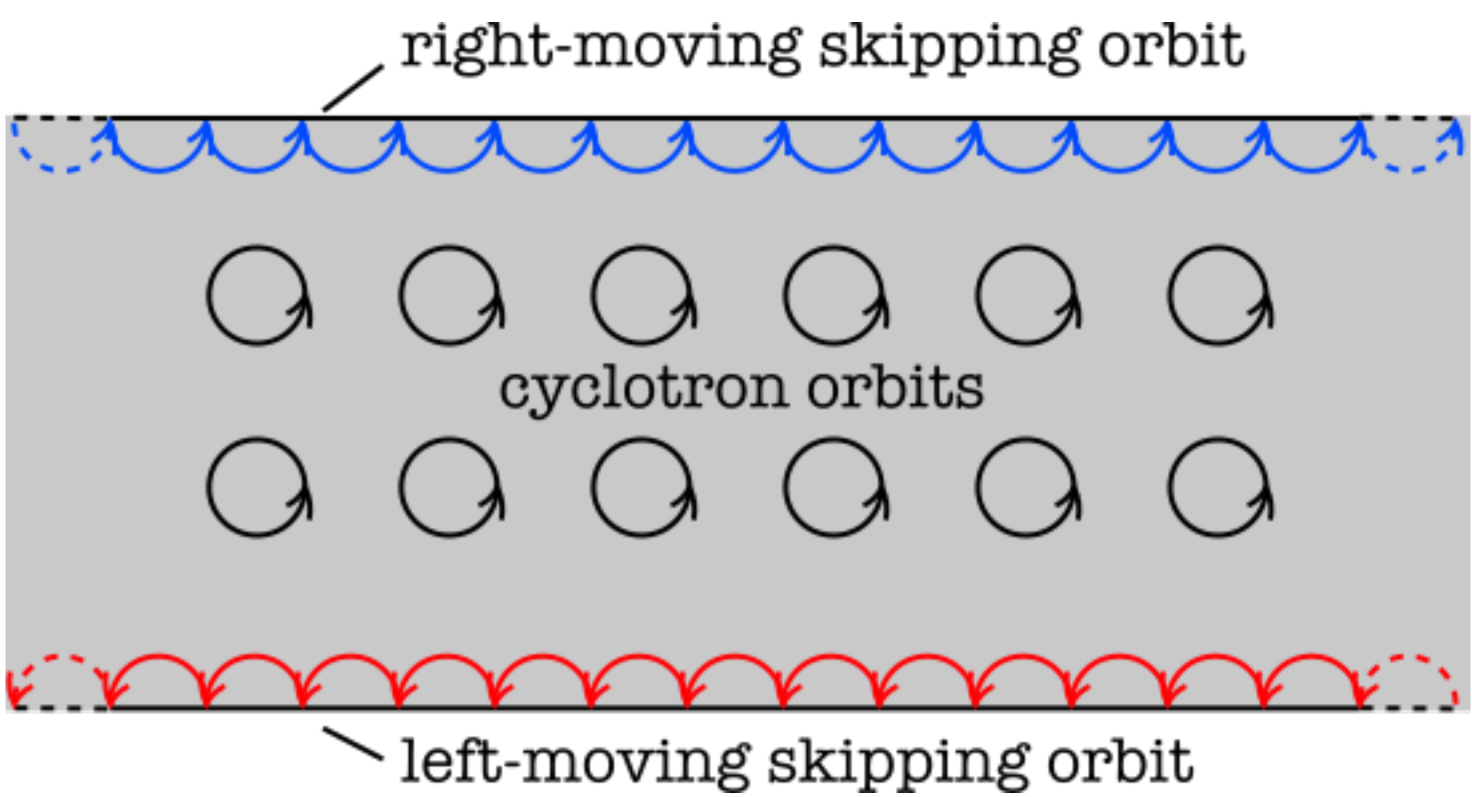
Chiral Fermion in Condensed Matter

PHYSICAL REVIEW B VOLUME 25, NUMBER 4 15 FEBRUARY 1982

Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential

B. I. Halperin
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

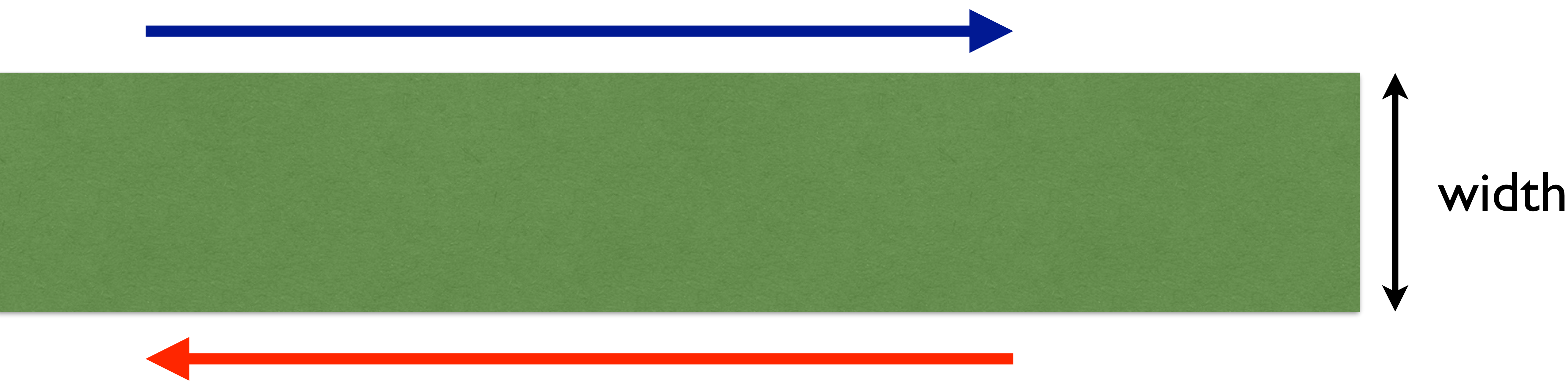
(Received 21 August 1981)



Chiral (Weyl) fermion in 1+1D as edge state

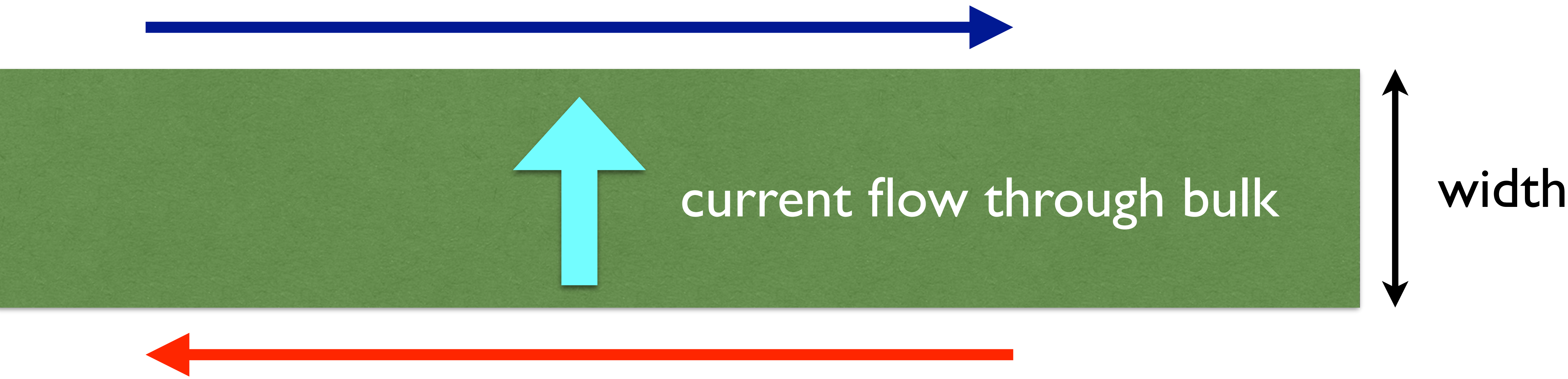
How did we avoid N-N theorem?

Nielsen-Ninomiya theorem applies to 1+1D system:
it does not apply in the limit of "infinite width"



For a finite width strip, Nielsen-Ninomiya theorem still applies,
and there indeed is a pair of left/right-moving Weyl fermion
which are spatially separated at the opposite edges

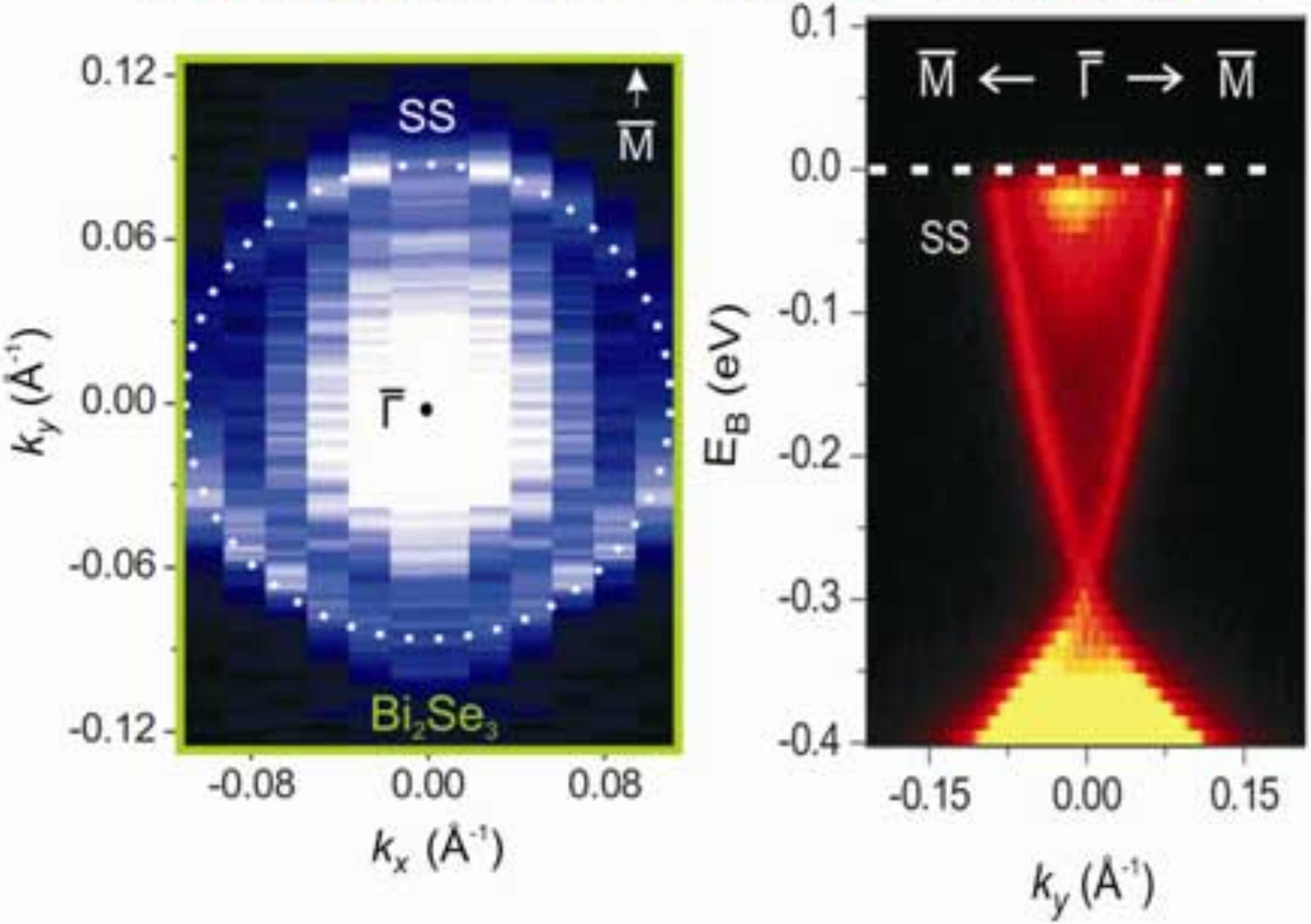
What about anomaly?



Anomalous field theory may be realized as
an edge/surface state of higher-dimensional lattice model

the “bulk” provides sink of anomalous current

Topological Insulator with a single Dirac cone



(Princeton University Group)

Anomaly & "No-Go" Theorems

Anomaly in quantum field theory implies
"no-go" theorem for lattice model
in the SAME DIMENSION

Chiral anomaly → Absence of chiral Dirac fermion on lattice
but 1+1D may be realized as
a chiral edge state of QHE in 2+1D

Parity anomaly → Absence of single massless Dirac fermion on
time-reversal invariant lattice model in 2+1D
but may be realized as a surface state of
a TR-invariant topological insulator in 3+1D

anomalous field theory may be realized at the edge/surface!

Anomaly in Interacting Systems

So far, I have discussed only non-interacting fermions

However, anomalies are believed to persist even in the presence of interactions

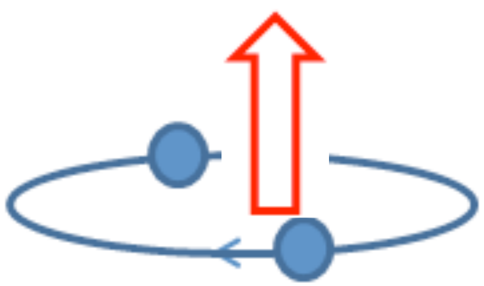
"Anomalous field theory may be only realized at the edge of a topological phase"*

Conversely, "anomalous field theory realized at the edge implies a topological phase in the (higher dimensional) bulk"*

should be still valid in interacting systems!

*: many caveats!

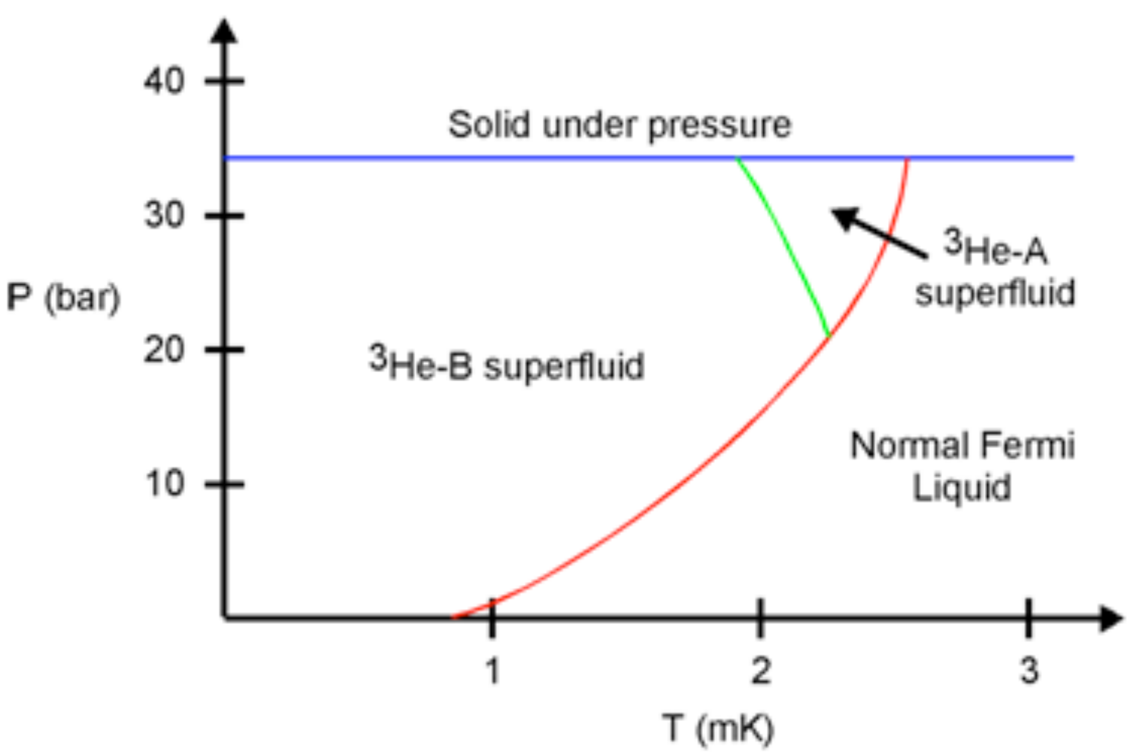
Chiral Superfluid



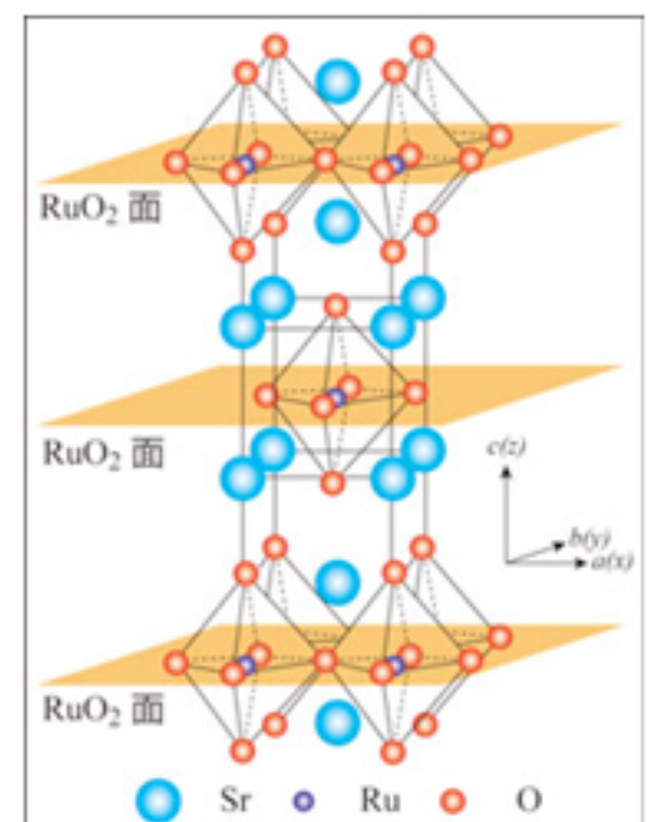
Cooper pair with
definite angular momentum $l_z=\nu$

pairing amplitude $\Delta \sim (p_x + ip_y)^\nu$

A-phase of superfluid ^{3}He



Superconducting phase of Sr_2RuO_4?



Chiral Majorana Edge State

Chiral p+ip superconductor in 2+1 D
has edge state which is chiral Majorana fermion in 1+1D

Chiral Majorana fermion is anomalous
$\rightarrow$ stable against perturbations
(No backscattering = "ingappable")

Stability of the edge state implies the topological nature of
the chiral p+ip superconductor in 2+1D
"topological superconductor"

Non-chiral edge state?

N_f copies of right-moving AND left-moving chiral Majorana fermions

$$\mathcal{L} = \frac{1}{4\pi} \sum_{a=1}^{N_f} \left[\psi_L^a (\partial_\tau + i v \partial_x) \psi_L^a + \psi_R^a (\partial_\tau - i v \partial_x) \psi_R^a \right].$$

equivalent to $N = N_f/2$ right/left-moving complex fermions

Can this be gapped by an edge perturbation?

Right-movers: from "↑ spin" Left-movers: from "↓ spin"

Symmetries: $n_\uparrow$ & $n_\downarrow$ separately conserved modulo 2 ⇒ $Z_2 \times Z_2$

mass term $\psi_L^a \psi_R^b$ is forbidden by this symmetry

⇒ non-interacting system is stable for any N_f

"Z classification"

Effect of Interactions?

Let us see if the edge theory is anomalous or not
(anomaly should give a criterion applicable to interacting systems)

[Ryu-Zhang, 2012]

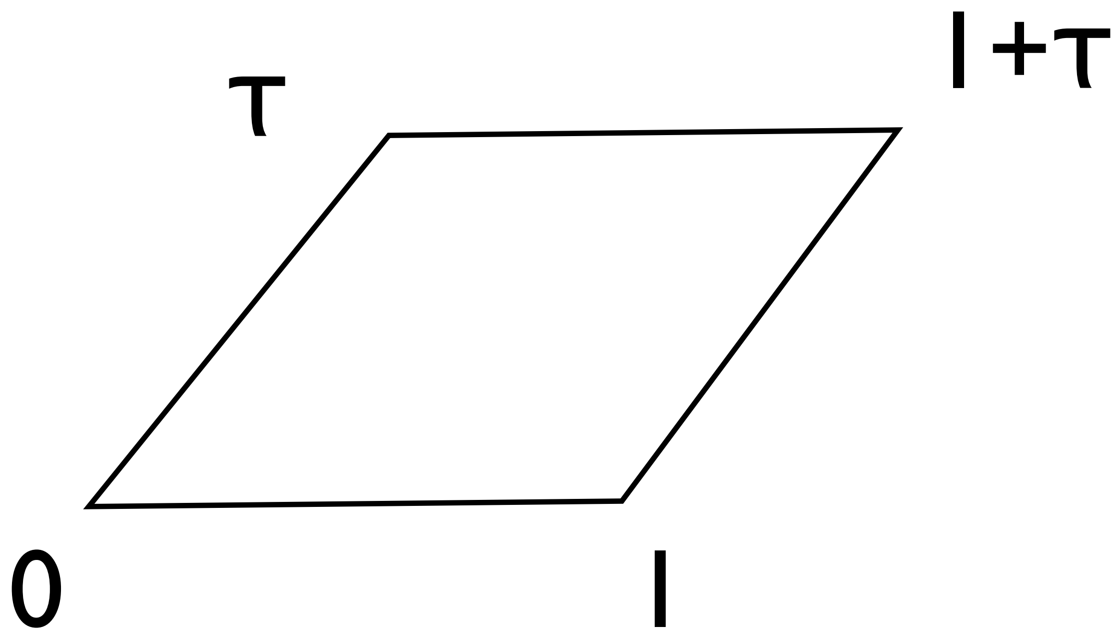
Impose the $Z_2 \times Z_2$ symmetry by "gauging"
or equivalently "orbifolding" (more on this later….)

$$P_{\mathrm{GSO}} = \frac{1+(-1)^{n_\uparrow}}{2} \frac{1+(-1)^{n_\downarrow}}{2}$$

$$Z_{\mathrm{orb}} \sim \mathrm{Tr}_A \left(P_{\mathrm{GSO}} e^{-\beta H_A} \right) + \mathrm{Tr}_P \left(P_{\mathrm{GSO}} e^{-\beta H_P} \right)$$

$$\sim \left| \frac{Z_{++} + Z_{+-} + Z_{-+} + Z_{--}}{2} \right|^2 \quad \text{(roughly)}$$

Modular Invariance



Partition function of a consistent CFT must be invariant under modular transformations generated by

$$\mathcal{S} : \tau \to -1/\tau \qquad \mathcal{T} : \tau \to \tau + 1$$

$$\mathcal{T}^2 : \tau \to \tau + 2 \quad \text{for fermions}$$

Single Complex Fermion

$$Z_{\lambda\mu}(\tau) = e^{2\pi i\lambda\mu} q^{-1/24} q^{\lambda^2/2} \prod_{n=1}^{\infty} (1 + wq^{n-1/2})(1 + w^{-1} q^{n-1/2})$$

$$q = e^{2\pi i\tau} \qquad \lambda, \mu = 0, 1/2 \qquad w = e^{2\pi i\mu} q^{\lambda}$$

$$Z_0^0(\tau + 1) = e^{-i\pi/12} Z_{1/2}^0(\tau)$$

$$Z_{1/2}^0(\tau + 1) = e^{-i\pi/12} Z_0^0(\tau)$$

$$Z_0^{1/2}(\tau + 1) = e^{i\pi/6} Z_0^{1/2}(\tau)$$

$$Z_{1/2}^{1/2}(\tau + 1) = e^{i\pi/6} Z_{1/2}^{1/2}(\tau)$$

Orbifold partition function of the single complex fermion (N_f =2, N=1) cannot be modular invariant

N=4 Complex Fermions

$$\left(Z_0^0(\tau+1)\right)^4 = e^{-i\pi/3}\left(Z_{1/2}^0(\tau)\right)^4$$

$$\left(Z_{1/2}^0(\tau+1)\right)^4 = e^{-i\pi/3}\left(Z_0^0(\tau)\right)^4$$

$$\left(Z_0^{1/2}(\tau+1)\right)^4 = e^{2i\pi/3}\left(Z_0^{1/2}(\tau)\right)^4$$

$$\left(Z_{1/2}^{1/2}(\tau+1)\right)^4 = e^{2i\pi/3}\left(Z_{1/2}^{1/2}(\tau)\right)^4$$

Chiral orbifold partition function is modular covariant if
N is an integral multiple of 4 (N_f is an integral multiple of 8)

Total (non-chiral) orbifold partition function

$Z_{\rm orb}(\tau,\bar\tau) = Z_{\rm orb}(\tau) Z_{\rm orb}(\bar\tau)$ is then modular invariant

Non-chiral edge state?

N_f copies of right-moving AND left-moving chiral Majorana fermions

$$\mathcal{L} = \frac{1}{4\pi} \sum_{a=1}^{N_f} \left[\psi_L^a (\partial_\tau + i v \partial_x) \psi_L^a + \psi_R^a (\partial_\tau - i v \partial_x) \psi_R^a \right].$$

Can this be gapped by an edge perturbation?

Right-movers: from "↑ spin"   Left-movers: from "↓ spin"

Symmetries: $n_\uparrow$ & $n_\downarrow$ separately conserved modulo 2 ⇒ $Z_2 \times Z_2$

non-interacting system is stable for any N_f "Z classification"

interacting system can be gapped only if N_f is an integral multiple of 8

"Z_8 classification"

cf.) Fidkowski-Kitaev 2010

Anomaly in Interacting Systems

So far, I have discussed only non-interacting fermions

However, anomalies are believed to persist even in the presence of interactions

“Anomalous field theory may be only realized at the edge of a topological phase”*

Conversely, “anomalous field theory realized at the edge implies a topological phase in the (higher dimensional) bulk”*

should be still valid in interacting systems!

***: many caveats!**

SU(2) WZW theories

Lorentz-invariant critical point:
expect chiral SU(2) x SU(2) symmetry

Natural action with the SU(2) x SU(2) symmetry

$$S_0 = \frac{1}{2\lambda^2}\int d^2x \; \mathrm{Tr}[(g^{-1}\partial_\mu g)^2]$$

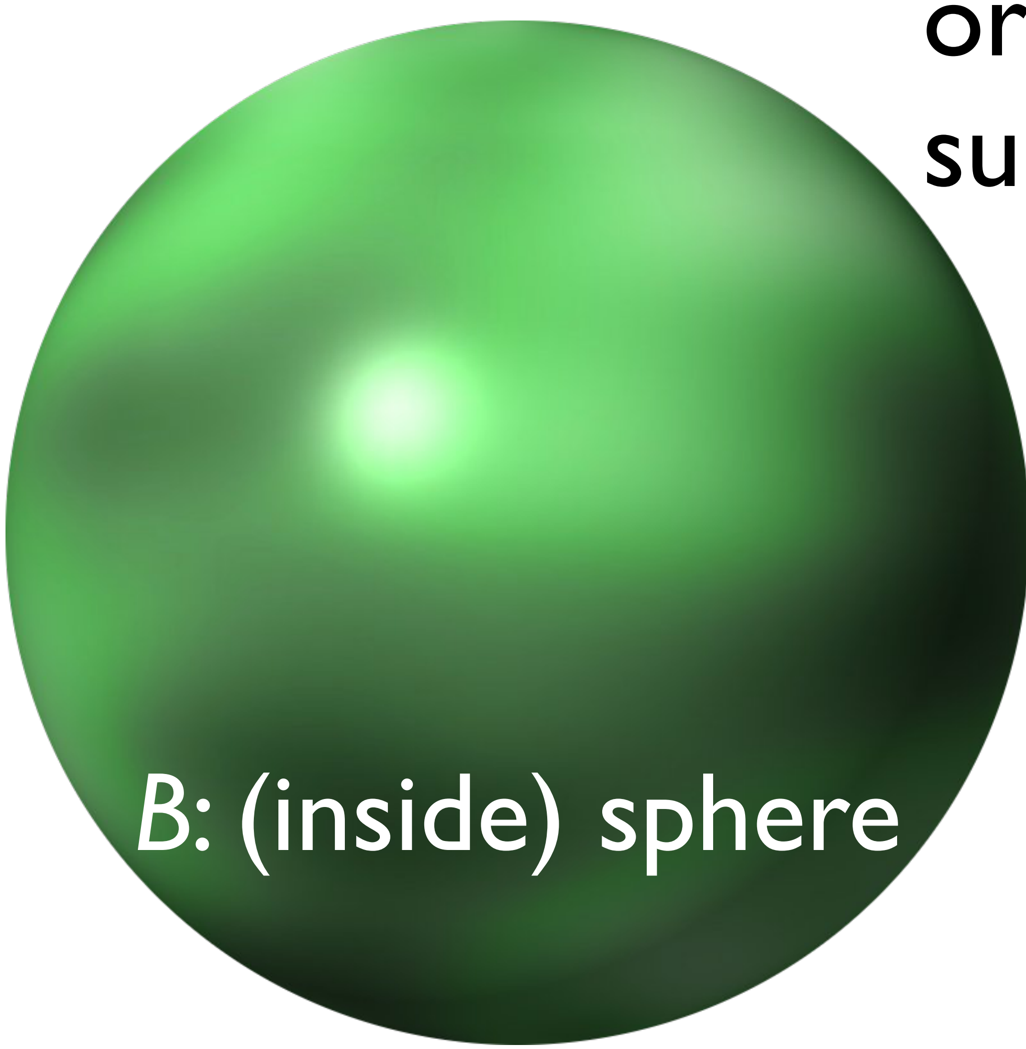
g: SU(2) matrix-valued field

However, RG implies that this theory is
always massive (gapped) "asymptotic freedom"

Wess-Zumino term

$$S = S_0 + k\Gamma_{WZ}$$

$$\Gamma_{WZ} = \frac{1}{12\pi}\int_B d^3x\ \epsilon^{ijk}\mathrm{Tr}[(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)]$$



original space-time:
surface of the sphere

uniqueness of $k\Gamma_{WZ}$
(modulo 2π)
$\Rightarrow$ ***k*: integer**

RG has a nontrivial fixed point
if $k \neq 0$ → gapless critical phase

Kac-Moody algebra

$$J(z) = \frac{k}{2} g^{-1} \partial_z g = \sum_n \frac{1}{z^{n+1}} J_n^a \frac{\sigma^a}{2}$$

$$[J_n^a, J_m^b] = i f_{abc} J_{n+m}^c + \frac{1}{2} k n \delta_{ab} \delta_{n+m,0}$$

This "includes" Virasoro algebra (conformal invariance) and is very powerful — determines scaling dimensions (critical exponents) etc.

$$c = \frac{3k}{k+2} \qquad h_j = \frac{j(j+1)}{k+2} \qquad 0 \leq j \leq \frac{k}{2}$$

central charge

scaling dimension of spin-*j* field

Discrete Symmetry

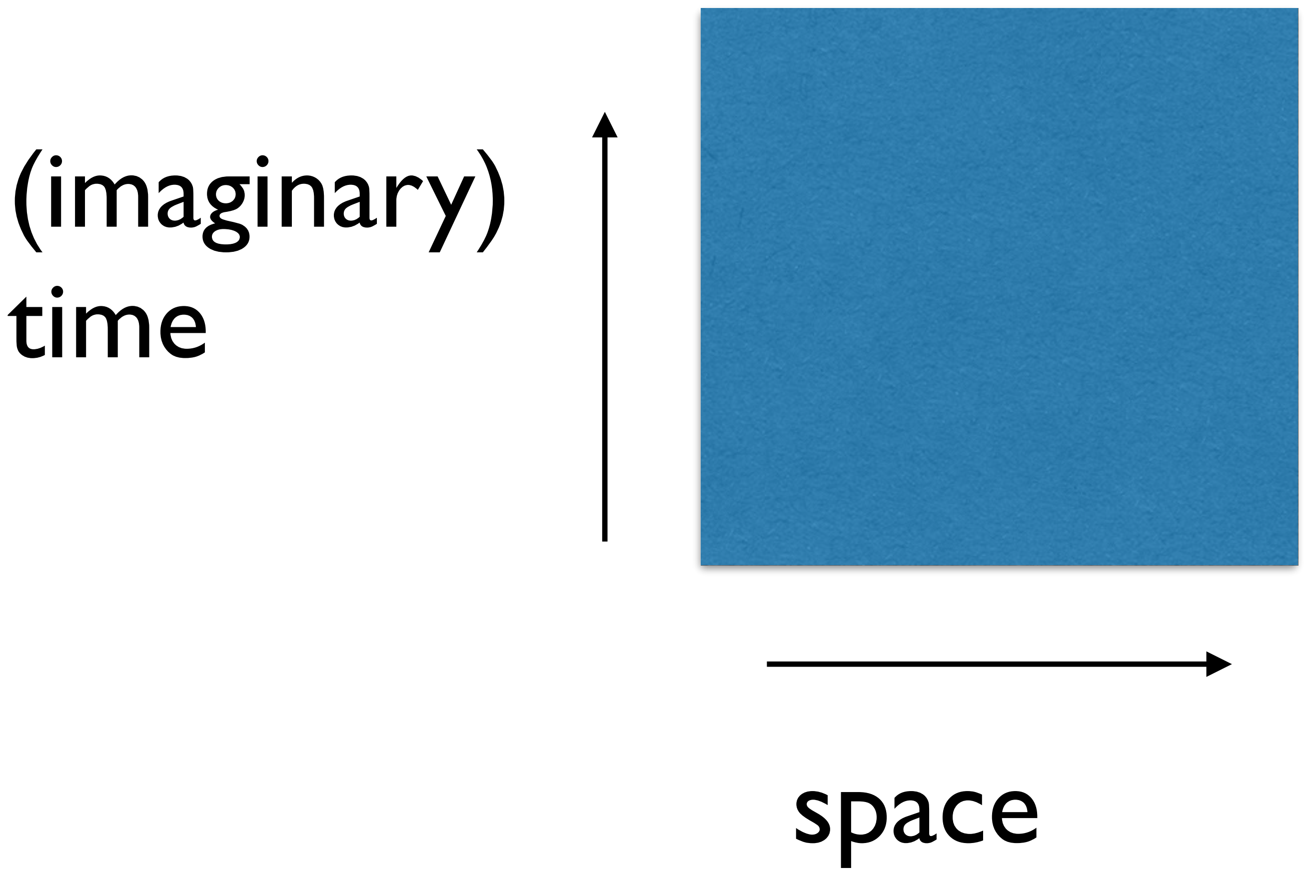
WZW action is also invariant under

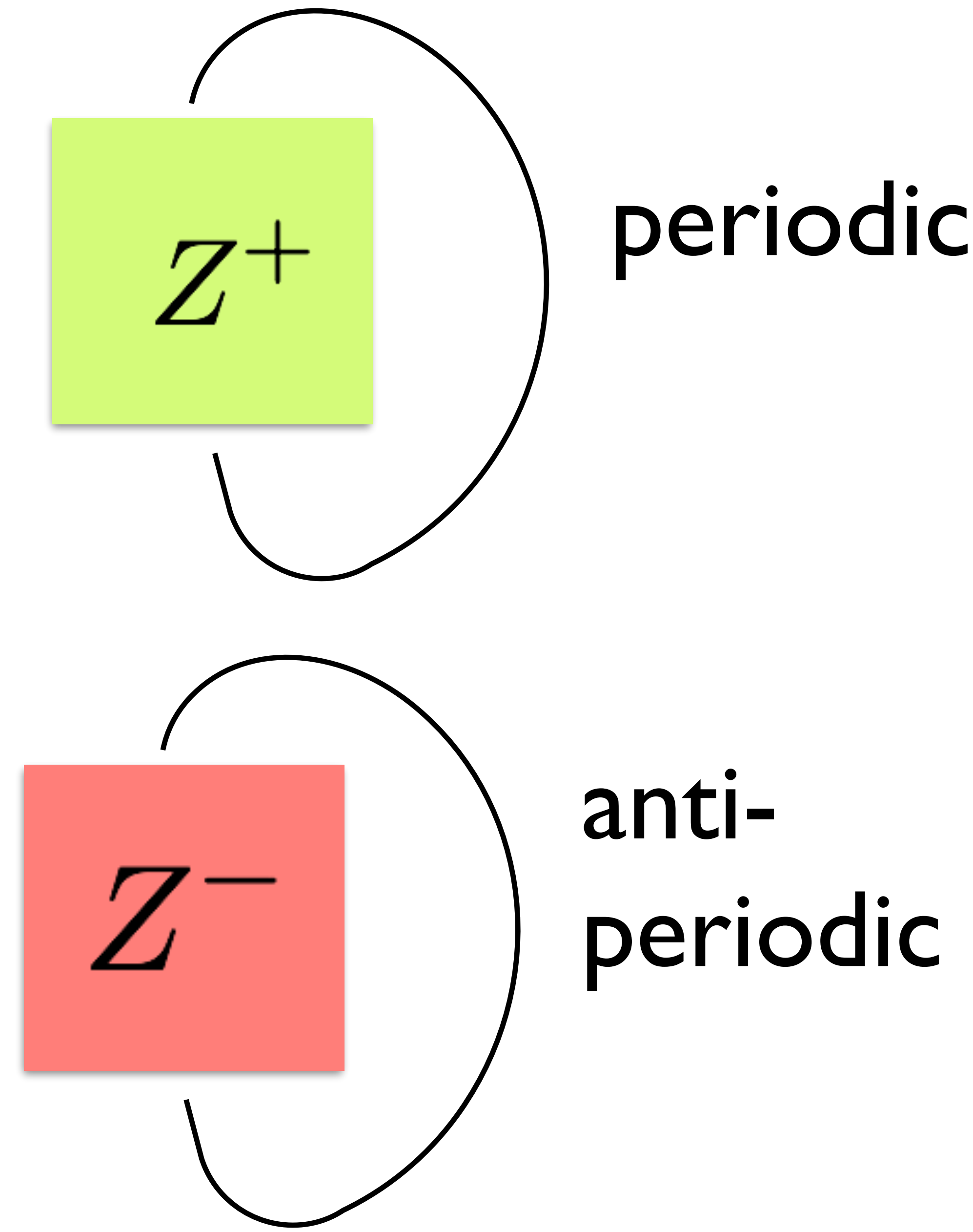
$$g \to -g$$

Discrete Z2 symmetry

Let us also consider gauging this Z_2 symmetry by considering the Z_2 orbifold….

Projection vs. Path Integral





$$Z_{+}^{\mathrm{proj}} = \mathrm{Tr}[P_{+} e^{-\mathcal{H}}] = \frac{1}{2}[Z^{+} + Z^{-}]$$

Orbifold Construction

The "projected" partition function Z_+^{proj} is not modular invariant by itself — must be supplemented by twisted sectors

$$Z_+ = (1 + \mathcal{S} + \mathcal{T}\mathcal{S}) Z_+^{\text{proj}} - Z_{\text{WZW}}$$

The resulting partition function represents the "Z_2 orbifold" of the original $SU(2)_k$ WZW theory

Global Anomaly

The Z_2 orbifold should be modular invariant by construction — but this is NOT always the case!

The Z_2 orbifold is **modular invariant if *k* is even**, but it is **modular NON-invariant if *k* is odd**

Gepner-Witten 1986

STRING THEORY ON GROUP MANIFOLDS

Doron GEPNER and Edward WITTEN

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA

Received 26 May 1986

What does it mean?

SU(2) WZW with an odd level k has the global Z2 anomaly
⇒ it can be only realized at the edge of a 2+1D topological phase?

However, it is known that
exactly solvable SU(2)-invariant spin chain with S=2k
(Takhtajan-Babujan model)
realizes the $SU(2)_k$ WZW , with the Z_2 symmetry,
in the low-energy limit!

Why the "anomalous" field theory can be realized in the lattice model in the same dimension?

What is the implication of the anomaly in this case?

Types of Anomaly

- ABJ anomaly
- t'Hooft anomaly

obstruction of gauging a global symmetry

orbifold construction = gauging the Z2 symmetry

the WZW theory is consistent for any integer k
if it is not gauged

SU(2)k with odd k has a t'Hooft anomaly concerning
the Z2 symmetry

Anomaly in Interacting Systems

So far, I have discussed only non-interacting fermions

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"Anomalous field theory may be only realized at the edge of a topological phase"*

Conversely, "anomalous field theory realized at the edge implies a topological phase in the (higher dimensional) bulk"*

should be still valid in interacting systems!

***: many caveats!**

Realization of anomalous theory

If a global symmetry of the field theory is realized
as a “on-site” symmetry of a lattice model,
it can be gauged exactly
$\Rightarrow$ a field theory with ’t Hooft anomaly cannot be realized
on a lattice

However, the global symmetry of the field theory
may be realized in other manners…

Spin chain and WZW

$$\vec{S}_i \sim \vec{J}_i + \mathrm{const.}(-1)^i \mathrm{tr}(g\vec{\sigma})$$

Lattice translation symmetry

$\Leftrightarrow$ discrete Z_2 symmetry $\quad g \to -g$

Translation symmetry of the lattice may not be gauged

Field theory with a 't Hooft anomaly may have a lattice realization in the same dimension, if the anomalous global symmetry corresponds to the translation symmetry

[Cho-Hsieh-Ryu 2017]

What does this mean?

If the orbifold is modular invariant, we can consider projection onto the symmetric sector, and open a gap within that sector to obtain the unique ground state

However, if it is modular non-invariant (ie. k: odd),
we cannot open the gap to obtain a unique ground state within the symmetric sector;
ground states in the symmetric/antisymmetric sectors must be degenerate!

"Lieb-Schultz-Mattis (LSM) theorem" from field theory

Lieb-Schultz-Mattis theorem

For **translation** & SU(2) invariant spin chains

if S is integer: no constraint

if S is half-odd-integer:
the system must be gapless,
OR the ground state is at least doubly degenerate

Lieb-Schultz-Mattis 1961 (S=1/2 chain at zero magnetization)
Affleck-Lieb 1986 (arbitrary S chain at zero magnetization)
MO-Yamanaka-Affleck 1997, MO 2000, Hastings 2004, etc etc.

more generally, "filling-enforced constraints"

Proof by large gauge invariance

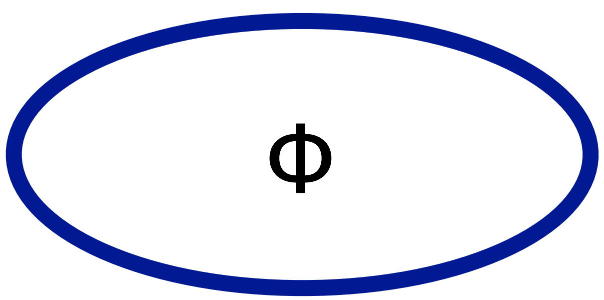
$$T_x|\Psi_0\rangle = e^{iP_x^0}|\Psi_0\rangle$$

LSM, Affleck-Lieb, M.O……

$$|\Psi_0\rangle \xrightarrow[\text{momentum unchanged}]{\text{adiabatic flux insertion}} |\Psi_0'\rangle$$

$$\Big\Downarrow \ \text{Large gauge transformation}$$

$$U_x|\Psi_0'\rangle$$



$$T_x\left(U_x|\Psi_0'\rangle\right) = e^{2\pi i\nu}U_xT_x|\Psi_0'\rangle$$
$$= e^{i(P_x^0+2\pi\nu)}\left(U_x|\Psi_0'\rangle\right)$$

momentum shift by $\quad 2\pi\nu = 2\pi(S-m)$

Letters in Mathematical Physics **12** (1986) 57–69.

Affleck-Lieb 1986
S: half-odd-integer
→ gapless or
2-fold g.s. degeneracy

A Proof of Part of Haldane's Conjecture on Spin Chains

IAN AFFLECK★ and ELLIOTT H. LIEB★★
Departments of Mathematics and Physics, Princeton University, P.O. Box 708, Princeton, NJ 08544, U.S.A.

(Received: 10 March 1986)

Abstract. It has been argued that the spectra of infinite length, translation and $U(1)$ invariant, anisotropic, antiferromagnetic spin s chains differ according to whether s is integral or $\frac{1}{2}$ integral: There is a range of parameters for which there is a unique ground state with a gap above it in the integral case, but no such range exists for the $\frac{1}{2}$ integral case. We prove the above statement for $\frac{1}{2}$ integral spin. We also prove that for *all s*, *finite length* chains have a unique ground state for a wide range of parameters. The argument was extended to $SU(n)$ chains, and we prove analogous results in that case as well.

was a generalization of
"Lieb-Schultz-Mattis Theorem"

ANNALS OF PHYSICS: **16**, 407–466 (1961)

Two Soluble Models of an Antiferromagnetic Chain

ELLIOTT LIEB, THEODORE SCHULTZ, AND DANIEL MATTIS
Thomas J. Watson Research Center, Yorktown, New York

II. THE XY MODEL

A. Formulation

The first model consists of N spin $\frac{1}{2}$'s (N even) arranged in a row and having only nearest neighbor interactions. It is

$$H_\gamma = \sum_i [(1 + \gamma) S^x{}_i S^x{}_{i+1} + (1 - \gamma) S^y{}_i S^y{}_{i+1}], \tag{2.1}$$

a's and $a^\dagger$'s do not preserve this mixed set of canonical rules. However, it is possible to transform to a new set of variables that are strictly Fermi operators and in terms of which the Hamiltonian is just as simple.[1] Let

$$c_i \equiv \exp\left[\pi i \sum_1^{i-1} a_j{}^\dagger a_j\right] a_i$$

Main Result of "LSM" paper:
S=1/2 XY chain is solvable
by mapping to fermions

What about the LSM theorem?

APPENDIX B. NONDEGENERACY OF THE GROUND STATE AND ABSENCE OF AN ENERGY GAP IN THE HEISENBERG MODEL

We prove two exact theorems about the ground state and excitation spectrum for a Heisenberg model with nearest neighbor interactions in one dimension.

Appendix….

APPENDIX B. NONDEGENERACY OF THE GROUND STATE AND ABSENCE OF AN ENERGY GAP IN THE HEISENBERG MODEL

We prove two exact theorems about the ground state and excitation spectrum for a Heisenberg model with nearest neighbor interactions in one dimension. The generalization to longer range interactions and higher-dimensional lattices is indicated. A further generalization to particles of spin $\neq \frac{1}{2}$ and a discussion of the ordering of excited state energy levels has been submitted for publication in the *Journal of Mathematical Physics* by Lieb and Mattis.

?!

Perhaps refers to this paper

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 4 JULY-AUGUST 1962

Ordering Energy Levels of Interacting Spin Systems

ELLIOTT LIEB AND DANIEL MATTIS
Thomas J. Watson Research Center, International Business Machines Corporation, Yorktown Heights, New York
(Received October 6, 1961)

But no mention is actually made on the generalization of LSM theorem?!

Maybe….

LSM tried to generalize their theorem to general *S*,
but "failed" to prove it for integer *S*

So they scrapped the generalization and never published
(until Affleck-Lieb paper 25 years ago)

…. maybe missing the evidence of the "Haldane gap"??

Selection Rule

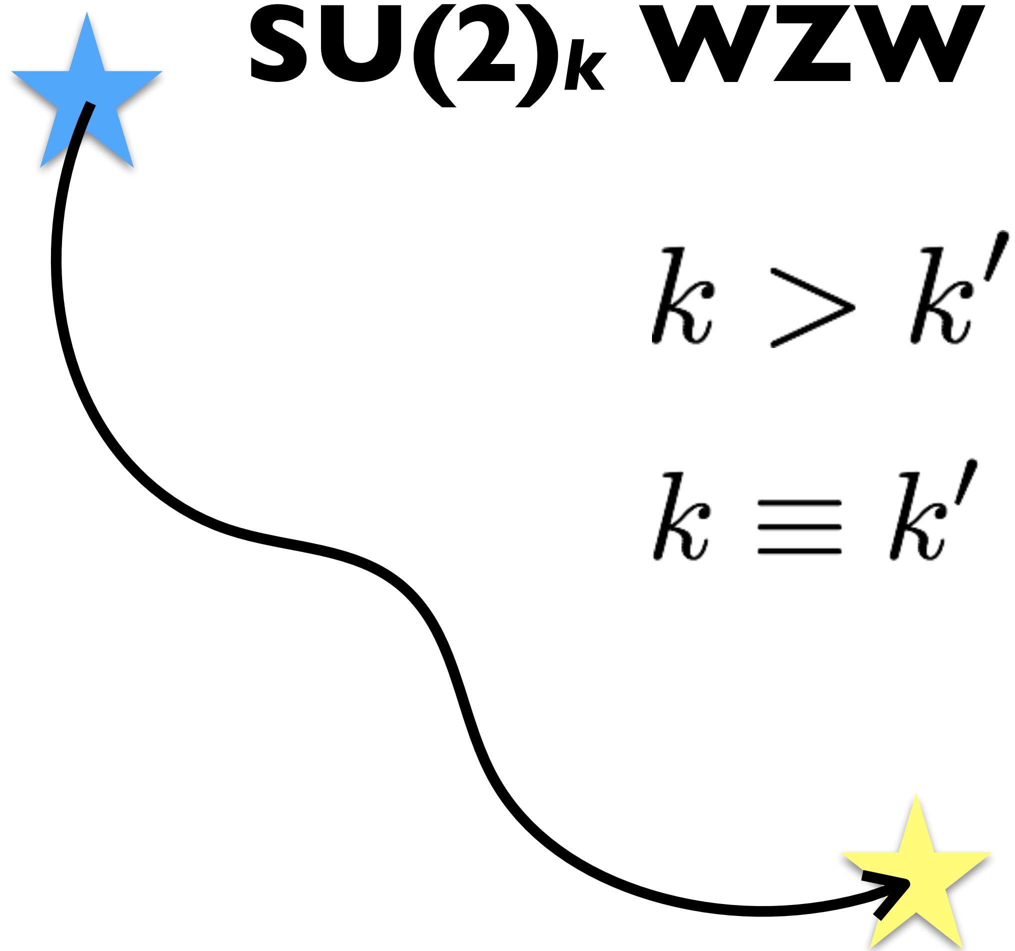
Perturb $SU(2)_k$ WZW with SU(2) and Z_2-symmetric relevant operators; suppose the RG flow reaches $SU(2)_{k'}$ WZW

if *k* is even, we should be able to consider the projection onto Z_2 symmetric sector; the RG flow can be understood in terms of the Z_2 orbifold → *k'* is also even

if *k* is odd, the IR fixed point should also have the global anomaly (otherwise contradicts with "LSM")
→ *k'* is also odd

"anomaly matching"

In terms of RG…



SU(2)$_k$ WZW

$k > k'$ ← c- theorem

$k \equiv k' \mod 2$ ← present work

SU(2)$_{k'}$ WZW

SU(2)$_0$ WZW is identified with gapped phase with a unique ground state

“Symmetry Protected” gapless phases

SU(2) + Lorentz + lattice translation symmetries

$SU(2)_k$ WZW	**$SU(2)_k$ WZW**
k: even	k: odd

Spin Chains and WZW

There is a special integrable (Bethe-ansatz solvable) spin chain model for any *S*,
Takhtajan-Babujian (TB) model

e.g. for *S*=1: $$\mathcal{H}_{TB} = \sum_j \left[\vec{S}_j \cdot \vec{S}_j - (\vec{S}_j \cdot \vec{S}_j)^2 \right]$$

Spin-*S* TB model is described by $SU(2)_{2S}$ WZW
(k=2S even if *S* is integer, k odd if *S* is half-odd integer)

Other models can be regarded as
TB model + perturbations, so
k: even if *S* is integer, k:odd if *S* is half-odd integer
if the one-site translation symmetry is kept

Our Claim

In the presence of the SU(2) and
lattice translation (by one site) symmetries,

$$S = 1/2, 3/2, 5/2, \ldots$$

- The system is gapped with a SSB of the translation symmetry (doubly degenerate GS)

OR - The system is gapless, described by **$SU(2)_k$ WZW with an odd k**

$$S = 1, 2, 3, \ldots$$

- The system is gapped (can be without SSB)

OR - The system is gapless, described by **$SU(2)_k$ WZW with an even k**

Anomaly and LSM

N_f Dirac fermions in 1+1D

[Cho-Hsieh-Ryu 2017]

$$H = \int dx \sum_{a=1}^{N_f} \left[\psi^\dagger_{L,a} i\partial_x \psi_{L,a} - \psi^\dagger_{R,a} i\partial_x \psi_{R,a} \right]$$

$$U(1)_{\delta\phi}: \quad \psi_{R,a}(x) \to e^{i\delta\phi q_a} \psi_{R,a}(x) \qquad \psi_{L,a}(x) \to e^{i\delta\phi q_a} \psi_{L,a}(x).$$

$$\mathbb{Z}_N: \quad \psi_{R,a}(x) \to e^{2\pi i s_{R,a}/N} \psi_{R,a}(x) \qquad \psi_{L,a}(x) \to e^{2\pi i s_{L,a}/N} \psi_{L,a}(x)$$

U(1)×Z_N symmetry ⇒ 't Hooft anomaly classified by

Spin^c cobordism group $\quad \Omega^3_{\mathrm{Spin}^c}(B\mathbb{Z}_N) \cong \mathbb{Z}_{\epsilon_N \cdot N} \times \mathbb{Z}_{N/\epsilon_N}$

$$\epsilon_N = \begin{cases} 1 & (N : \mathrm{odd}) \\ 2 & (N : \mathrm{even}) \end{cases}$$

Anomaly and LSM

[Cho-Hsieh-Ryu 2017]

$$H = \int dx \sum_{a=1}^{N_f} \left[\psi^\dagger_{L,a} i\partial_x \psi_{L,a} - \psi^\dagger_{R,a} i\partial_x \psi_{R,a} \right]$$

$$U(1)_{\delta\phi}: \quad \psi_{R,a}(x) \to e^{i\delta\phi q_a} \psi_{R,a}(x) \qquad \psi_{L,a}(x) \to e^{i\delta\phi q_a} \psi_{L,a}(x).$$

$$\mathbb{Z}_N: \quad \psi_{R,a}(x) \to e^{2\pi i s_{R,a}/N} \psi_{R,a}(x) \qquad \psi_{L,a}(x) \to e^{2\pi i s_{L,a}/N} \psi_{L,a}(x)$$

$$\nu_a = \frac{s_{R,a} - s_{L,a}}{N}$$

$$\Omega^3_{\mathrm{Spin}^c}(B\mathbb{Z}_N) \cong \mathbb{Z}_{\epsilon_N \cdot N} \times \mathbb{Z}_{N/\epsilon_N}$$

$$\sum_a \nu_a \frac{s_{R,a} + s_{L,a}}{\epsilon_N} \bmod \mathbb{Z}, \qquad \sum_a \nu_a q_a \bmod \mathbb{Z}$$

Chiral Anomaly, LSM

Summary

Anomaly:

symmetry of Lagrangian violated in quantization

$\Leftrightarrow$ emergent symmetry in condensed matter/lattice model

Exact requirement of "anomalous" symmetry often leads to "no-go theorem" on lattice realization (Nielsen-Ninomiya etc.)

However, an anomalous field theory can generally be realized as edge/boundary of a higher-dimensional condensed matter/lattice model

A field theory with 't Hooft anomaly may be realized in the same space-time dimensions, if the symmetry is not "on-site"

e.g. translation symmetry corresponds to 't Hooft anomaly

$\rightarrow$ field-theory version of "Lieb-Schultz-Mattis theorem"

symmetry protection of gapless, critical phases