AKLT and Beyond:

Ian Affleck's Contributions to 1D quantum many- body physics

Masaki Oshikawa
ISSP, University of Tokyo

30 years of AKLT: Interacting Systems in Low Dimensions
April 26-28, 2018 at UBC, Vancouver, Canada

VOLUME 59, NUMBER 7 PHYSICAL REVIEW LETTERS 17 AUGUST 1987

Rigorous Results on Valence-Bond Ground States in Antiferromagnets

Ian Affleck,[a] Tom Kennedy, Elliott H. Lieb, and Hal Tasaki
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 26 May 1987)

30 years of "AKLT"!

Commun. Math. Phys. 115, 477–528 (1988)

Communications in Mathematical Physics

Valence Bond Ground States in Isotropic Quantum Antiferromagnets

Ian Affleck[1,*], Tom Kennedy[2,**], Elliott H. Lieb[2,***], and Hal Tasaki[2,***]

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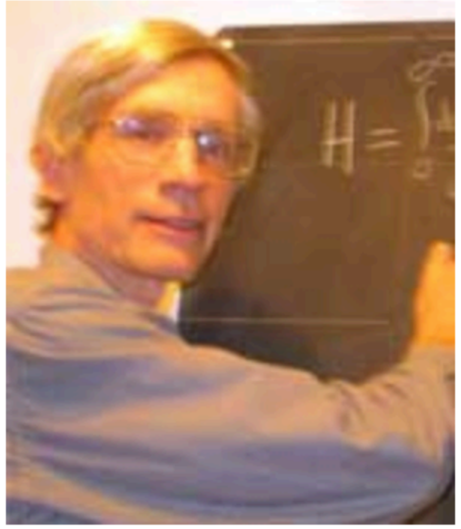
Home | Programs | APS Honors | Prizes | Lars Onsager Prize

2012 Lars Onsager Prize Recipient

Ian Affleck
University of British Columbia

to recognize outstanding research in theoretical statistical physics including the quantum fluids

Citation:



"For his pioneering role in developing and applying the ideas and methods of conformal field theory to important problems in statistical and condensed matter physics, including the quantum critical behavior of spin chains and (with Ludwig) the universal behavior of quantum impurity systems."

application of CFT including
quantum critical behavior of spin chains
and universal behavior of quantum impurity systems

→ talks by Eggert, Sirker, Giuliano, Pereira, …..

AKLT: non-critical (gapped) phase of spin chains
not included(!) in the citation of the Onsager Prize…

Haldane “Conjecture”

Heisenberg antiferromagnetic chain [Haldane 1981]

$$\mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

***S*=1/2, 3/2, 5/2........**

“critical”: gapless, power-law decay of spin correlations
(known for S=1/2 from Bethe Ansatz exact solution)

***S*=1, 2, 3, ...**

“disordered”: **non-zero gap**, **exponential decay of spin correlations**

against the “common sense” at that time

AKLT Model&State

$$\mathcal{H}_{\mathrm{AKLT}}^{S=1} = \sum_j \left[\vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} \left(\vec{S}_j \cdot \vec{S}_{j+1} \right)^2 \right]$$

$$= \sum_j 2 \left[P_2 \left(\vec{S}_j + \vec{S}_{j+1} \right) - \frac{2}{3} \right]$$

Exact ground state: AKLT (Valence-Bond-Solid, VBS) state

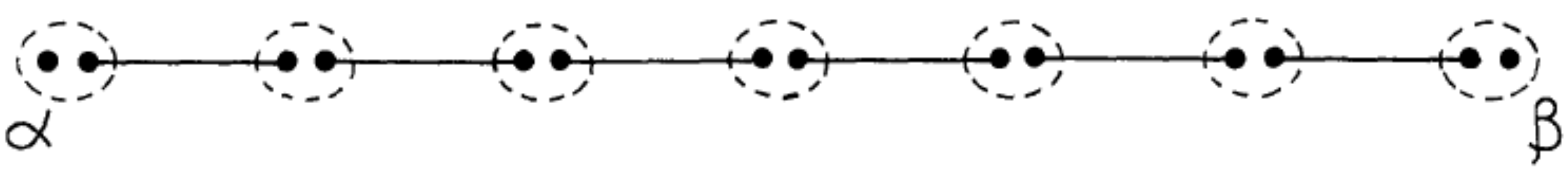


Fig. 2.1. The VBS state $\Omega_{\alpha\beta}$ on a finite chain. Each dot, line, and dotted circle represents a spin 1/2, a singlet pair, and the symmetrization of two spin 1/2's to create a spin 1

AKLT realizes Haldane “Conjecture”

Exact correlation function shows the exponential decay!

by (4). We find, for an infinite chain,

$$\langle S_0^a S_r^b \rangle = \delta^{ab}(-1)^r \tfrac{4}{3}\, 3^{-r} \text{ (for all } r > 0).$$

Thus, the correlation function decays exponentially with correlation length $1/\ln 3 \approx 0.9$.

Presence of the non-zero gap is proved!

2.3. The Energy Gap

We will prove that the Hamiltonian (2.2) has a gap between the ground state energy and the first excited state. In this subsection we begin by showing that for a finite chain of length L the gap is bounded away from zero as $L \to \infty$. In Sect. 2.4 we will show that the infinite volume system has a gap.

Exact/rigorous proof of Haldane “conjecture” (although for a modified model).

Is it just "disordered"?

Correction function of any (local) observable decays exponentially in the AKLT state

"quantum disordered"?

AKLT state is found to possess several "strange" properties

- S=1/2 "edge state" [Kennedy 1990]
- hidden antiferromagnetic (string) order [den Nijs-Rommelse 1989]

→ Hidden $Z_2 \times Z_2$ symmetry breaking [Kennedy-Tasaki 1992]

talks by Hagiwara & Tasaki

Topological Insulators

2005: theoretical discovery of "topological insulator"

PRL **95,** 226801 (2005) PHYSICAL REVIEW LETTERS week ending 25 NOVEMBER 2005

Quantum Spin Hall Effect in Graphene

C. L. Kane and E. J. Mele
Dept. of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA
(Received 29 November 2004; published 23 November 2005)

TI looks like an ordinary band insulator in the bulk,
but a nontrivial phase (e.g. with gapless edge states)
protected by a symmetry

Natural question:
what would be a generalization of the TI
to general, interacting many-body systems?

Symmetry-Protected Topological Phase

Selected for a Viewpoint in *Physics*

PHYSICAL REVIEW B **80**, 155131 (2009)

Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order

Zheng-Cheng Gu and Xiao-Gang Wen
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 8 March 2009; published 26 October 2009)

loop-gas model, and 1+1D quantum spin-1/2 and spin-1 models. In particular, using such a $(G_{\text{sym}}, T_{\text{inv}})$ characterization, we show that the Haldane phase for a spin-1 chain is a phase protected by the time-reversal, parity, and translation symmetries. Thus the Haldane phase is a symmetry-protected topological phase. The

PHYSICAL REVIEW B **81**, 064439 (2010)

Entanglement spectrum of a topological phase in one dimension

Frank Pollmann and Ari M. Turner
Department of Physics, University of California, Berkeley, California 94720, USA

Erez Berg
Department of Physics, Stanford University, Stanford, California 94305, USA
and Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

Masaki Oshikawa
Institute for Solid State Physics, University of Tokyo, Kashiwa 277-8581, Japan
(Received 9 October 2009; revised manuscript received 29 January 2010; published 26 February 2010)

PHYSICAL REVIEW B **85**, 075125 (2012)

Symmetry protection of topological phases in one-dimensional quantum spin systems

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[2]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
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[4]*Institute for Solid State Physics, University of Tokyo, Kashiwa 277-8581, Japan*
(Received 25 November 2011; published 22 February 2012)

We discuss the characterization and stability of the Haldane phase in integer spin chains on the basis of simple, physical arguments. We find that an odd-S Haldane phase is a topologically nontrivial phase which is protected by any one of the following three global symmetries: (i) the dihedral group of π rotations about the x, y, and z axes, (ii) time-reversal symmetry $S^{x,y,z} \to -S^{x,y,z}$, and (iii) link inversion symmetry (reflection about a bond center), consistent with previous results [Phys. Rev. B **81**, 064439 (2010)]. On the other hand, an even-S Haldane phase is not topologically protected (i.e., it is indistinct from a trivial, site-factorizable phase). We show some numerical evidence that supports these claims, using concrete examples.

What does “SPT phase” actually mean?

$\Rightarrow$ talk by Tasaki

AKLT state (for odd-*S* chain) already had characteristic features of a SPT phase!

It could have been possible to find the concept of SPT phases from AKLT states back in 1990s, much earlier than the discovery of TIs

However, it took 22 years after AKLT to reach the deeper understanding of the AKLT state, stimulated by TIs and other developments….

Matrix Product States

Original AKLT state: SU(2)-symmetric spin state

Generalizations?

Finitely Correlated States on Quantum Spin Chains

Commun. Math. Phys. 144, 443–490 (1992)

M. Fannes[1,2], B. Nachtergaele[3,4], and R. F. Werner[5]

EUROPHYSICS LETTERS

Europhys. Lett., **24** (4), pp. 293-297 (1993)

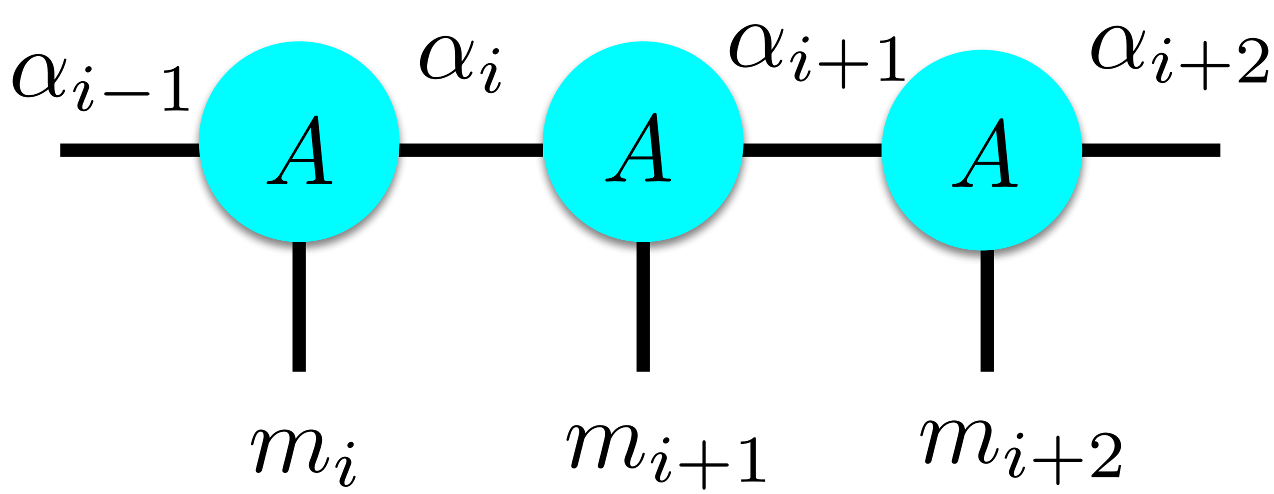
Matrix Product Ground States for One-Dimensional Spin-1 Quantum Antiferromagnets.

A. Klümper, A. Schadschneider and J. Zittartz

Matrix Product States

$$|\Psi\rangle = \sum_{\{\alpha_j\},\{m_j\}} \ldots A^{m_i}_{\alpha_{i-1}\alpha_i} A^{m_{i+1}}_{\alpha_i\alpha_{i+1}} A^{m_{i+2}}_{\alpha_{i+1}\alpha_{i+2}} \ldots |\ldots m_i m_{i+1} m_{i+2} \ldots\rangle$$

"virtual (bond) indices"



$$\alpha_j = 1, 2, \ldots, \chi$$

$$m_j = 1, \ldots, d$$

"physical indices"

Significance of MPS:

Any gapped ground state in 1D can be approximated by a MPS with a finite bond dimension χ

[Hastings 2007]

What does it mean?

Generic pure quantum state

$$|\Phi\rangle = \sum_{\{m_j\}} c_{\dots m_i m_{i+1} m_{i+2} \dots} | \dots m_i m_{i+1} m_{i+2} \dots \rangle$$

number of parameters: d^L

d: dimension of local Hilbert space

(Translation-invariant) MPS

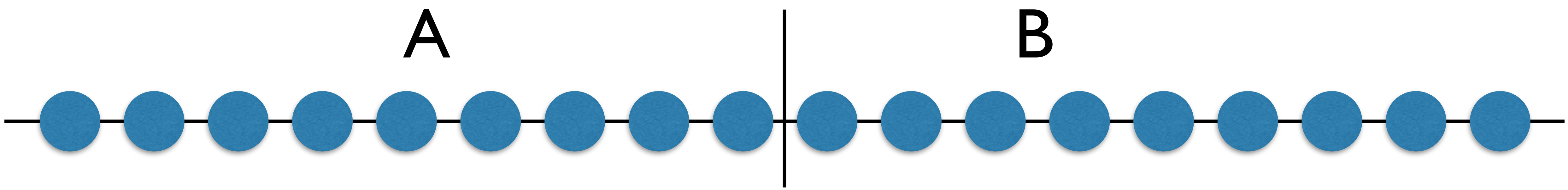
$$|\Psi\rangle = \sum_{\{\alpha_j\},\{m_j\}} \dots A^{m_i}_{\alpha_{i-1}\alpha_i} A^{m_{i+1}}_{\alpha_i \alpha_{i+1}} A^{m_{i+2}}_{\alpha_{i+1}\alpha_{i+2}} \dots | \dots m_i m_{i+1} m_{i+2} \dots \rangle$$

number of parameters: $d\chi^2 \ll d^L$!!

MPSs are very special among generic quantum states (yet can represent general gapped ground states in 1D!)

Why does MPS work??

Gapped ground states: finite correlation length



Bipartite entanglement entropy between A & B :

$S_E \sim \text{constant}$ ("Area Law")

even for an infinitely long system

Schmidt decomposition

$$|\Psi\rangle = \sum_{\gamma} \lambda_{\gamma} |\Psi_{\gamma}^{A}\rangle \otimes |\Psi_{\gamma}^{B}\rangle$$

need $O(e^{S_E}) \sim \text{const.}$ terms in the sum!

Successive Schmidt decompositions → MPS [Vidal 2003]

Applications…

Density-Matrix Renormalization Group (DMRG)

powerful numerical approach to 1D quantum many-body problems

[White 1992]

DMRG = variational methods with MPS

The density-matrix renormalization group in the age of matrix product states

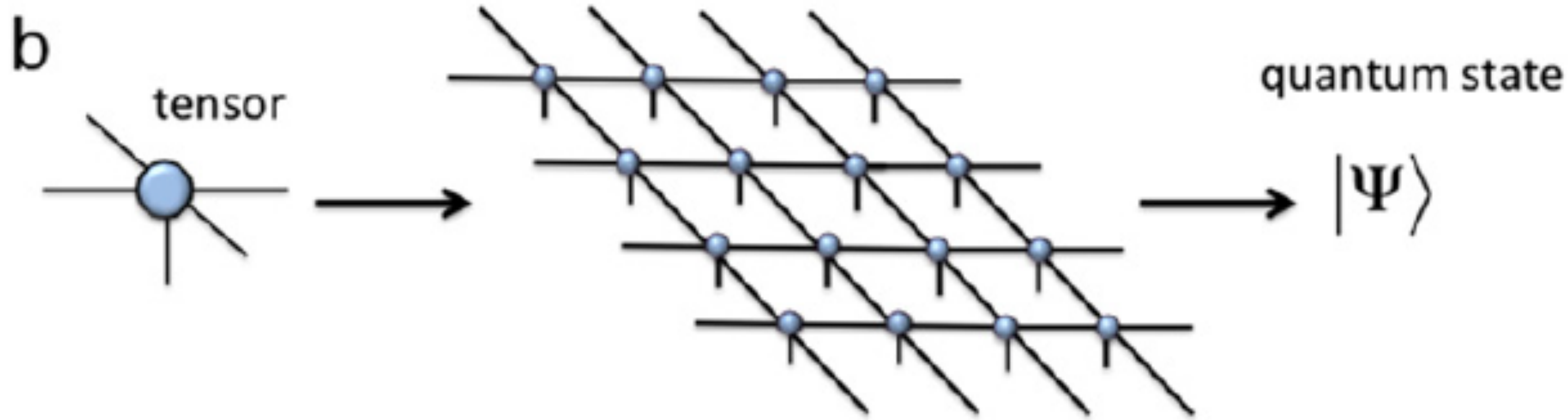
Ulrich Schollwöck

Arnold Sommerfeld Center for Theoretical Physics and Center for NanoScience, University of Munich, Theresienstrasse 37, 80333 Munich, Germany
Institute for Advanced Study Berlin, Wallotstrasse 19, 14159 Berlin, Germany

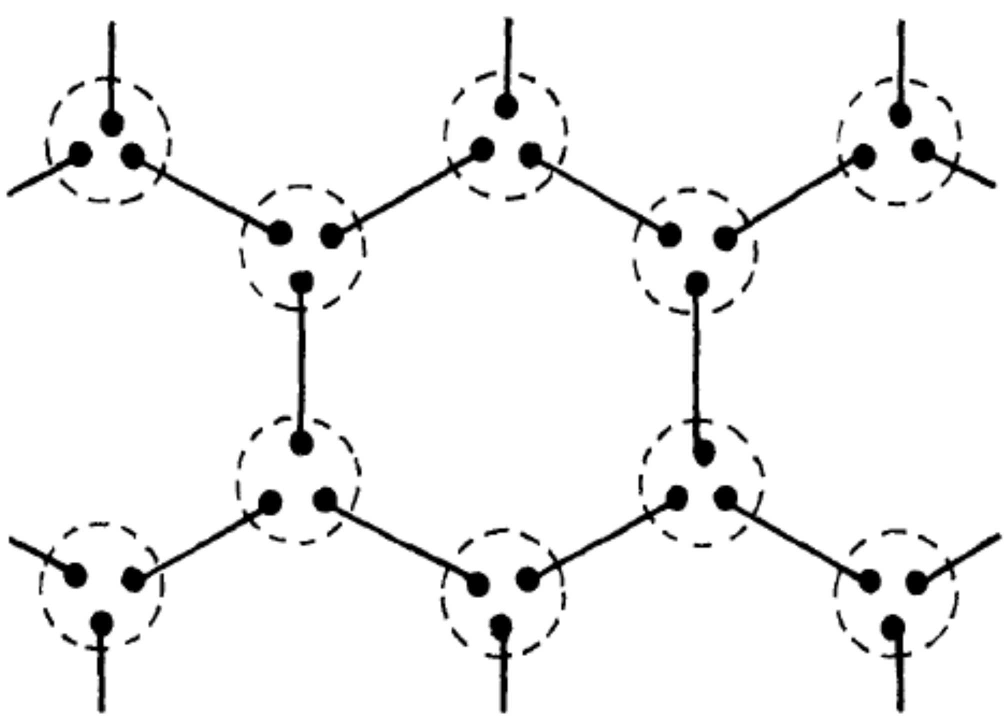
New perspectives, systematic improvements,…..

Higher Dimensions

Generalization of MPS: "tensor network states"



Who did it first??



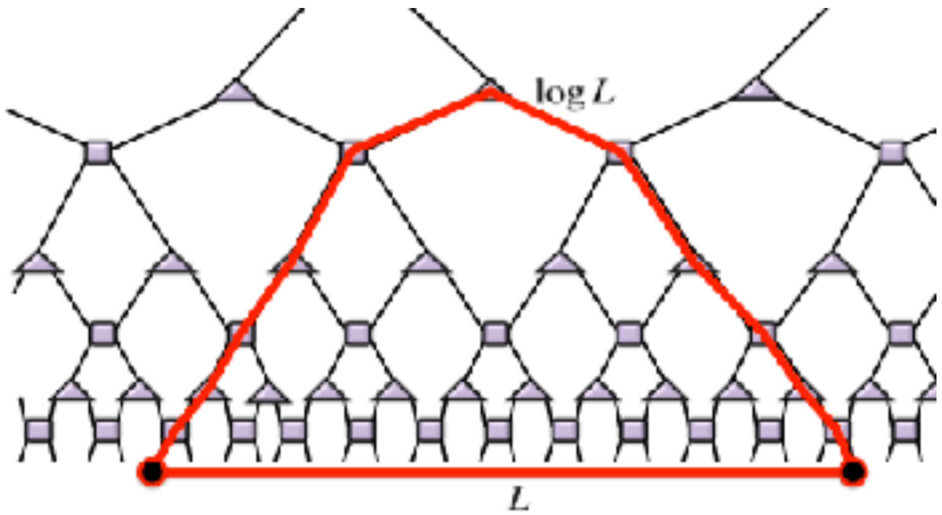
Valence Bond Ground States in Isotropic Quantum Antiferromagnets

Ian Affleck[1,⋆], Tom Kennedy[2,⋆⋆], Elliott H. Lieb[2,⋆⋆⋆], and Hal Tasaki[2,⋆⋆⋆]

[AKLT 1988]

Fig. 3.2. The VBS state on the hexagonal lattice. Each dot, line, and dotted circle represents a spin 1/2, a singlet pair, and the symmetrization of three spin 1/2's to create a spin 3/2

MERA

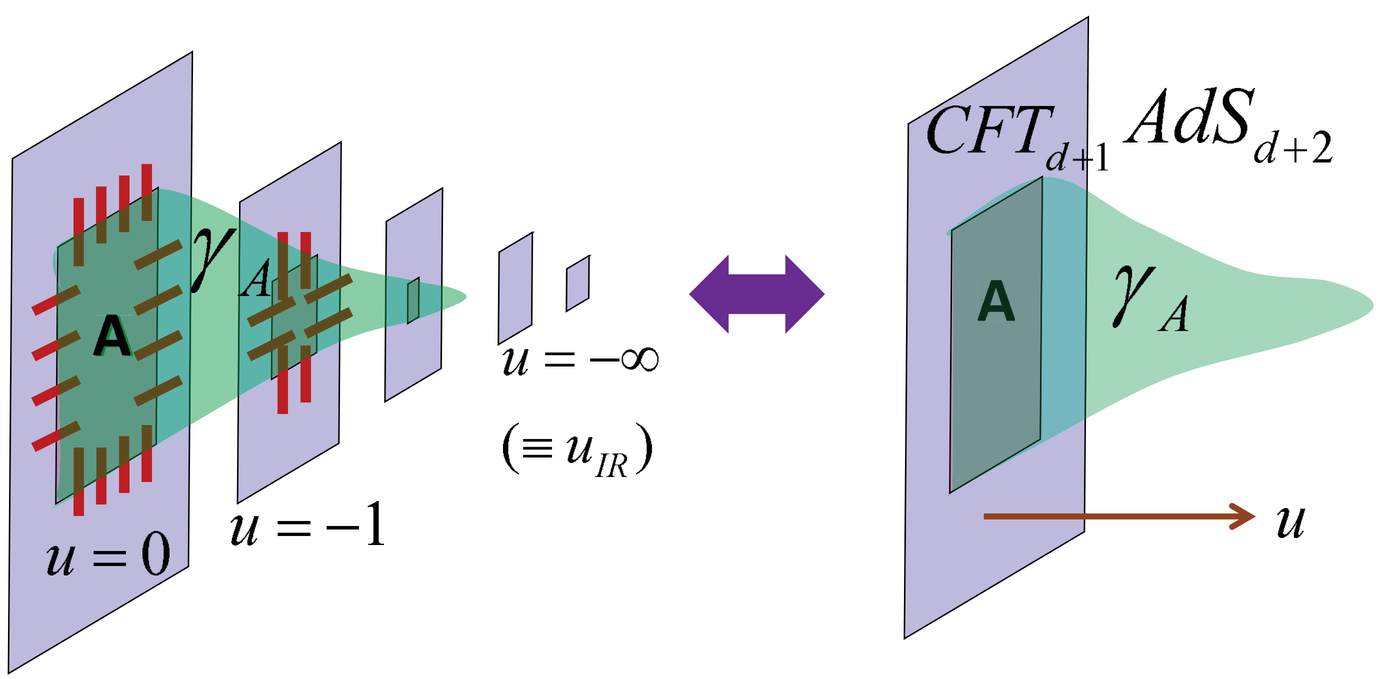


Multiscale Entanglement
Renormalization Ansatz
[Evenbly-Vidal 2009]

tree-like structure
for 1D critical states

$$S_E \propto \log L$$

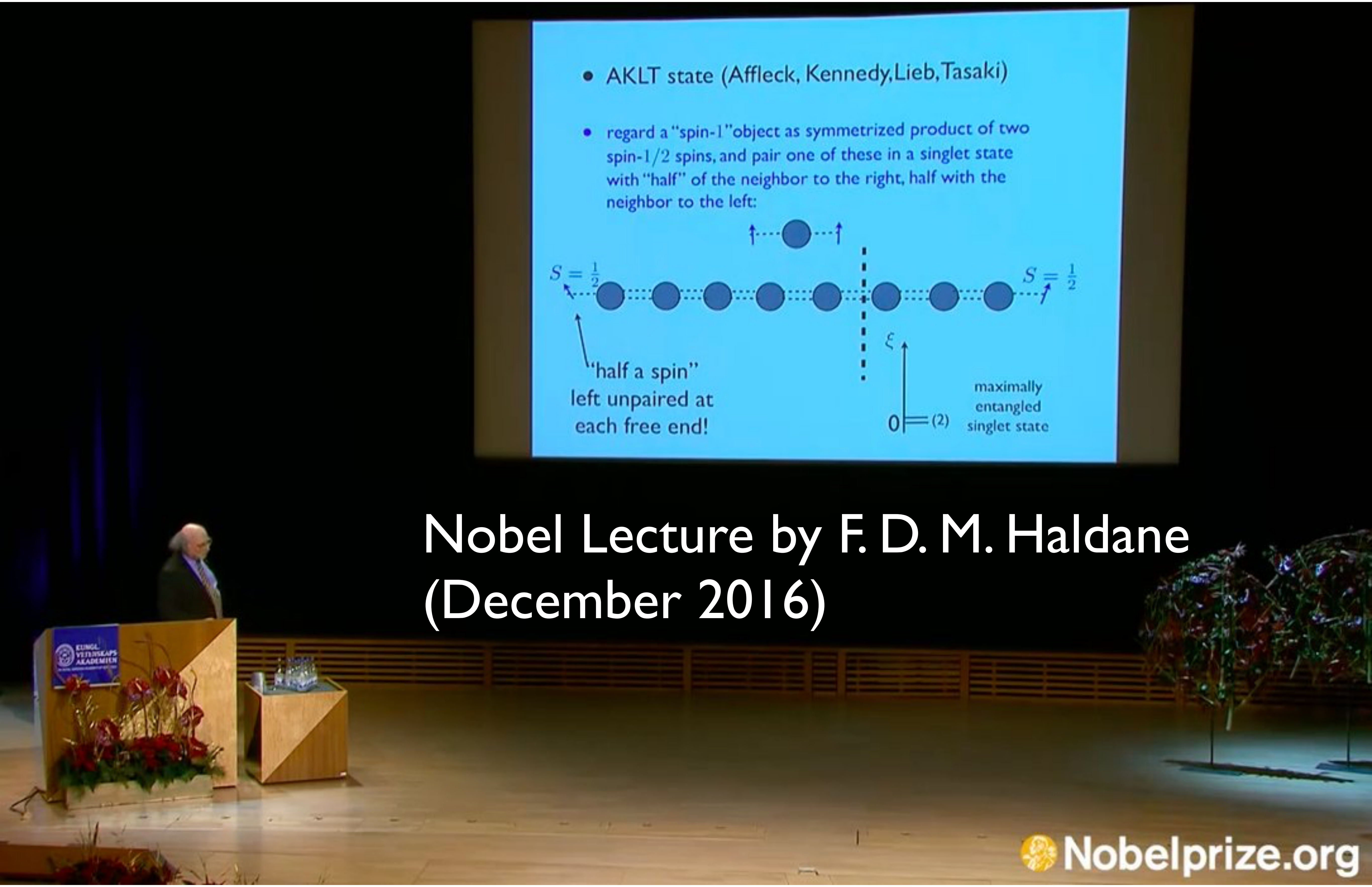
MERA AdS/CFT



$S_A \propto \mathrm{Min}[\#\mathrm{Bonds}(\gamma_A)]$ $S_A \propto \mathrm{Min}[\mathrm{Area}]$

Relation to
gauge-gravity duality
(AdS/CFT)

Figure from
[Nozaki-Ryu-Takayanagi 2012]



Nobel Lecture by F. D. M. Haldane (December 2016)

Letters in Mathematical Physics **12** (1986) 57–69.

Affleck-Lieb 1986

S: half-odd-integer

→ gapless or

2-fold g.s. degeneracy

A Proof of Part of Haldane's Conjecture on Spin Chains

IAN AFFLECK* and ELLIOTT H. LIEB**
Departments of Mathematics and Physics, Princeton University, P.O. Box 708, Princeton, NJ 08544, U.S.A.

(Received: 10 March 1986)

Abstract. It has been argued that the spectra of infinite length, translation and $U(1)$ invariant, anisotropic, antiferromagnetic spin s chains differ according to whether s is integral or $\frac{1}{2}$ integral: There is a range of parameters for which there is a unique ground state with a gap above it in the integral case, but no such range exists for the $\frac{1}{2}$ integral case. We prove the above statement for $\frac{1}{2}$ integral spin. We also prove that for *all* s, *finite length* chains have a unique ground state for a wide range of parameters. The argument was extended to $SU(n)$ chains, and we prove analogous results in that case as well.

was a generalization of

"Lieb-Schultz-Mattis Theorem"

ANNALS OF PHYSICS: **16**, 407–466 (1961)

Two Soluble Models of an Antiferromagnetic Chain

ELLIOTT LIEB, THEODORE SCHULTZ, AND DANIEL MATTIS
Thomas J. Watson Research Center, Yorktown, New York

II. THE XY MODEL

A. Formulation

The first model consists of N spin $\frac{1}{2}$'s (N even) arranged in a row and having only nearest neighbor interactions. It is

$$H_\gamma = \sum_i [(1 + \gamma) S^x_i S^x_{i+1} + (1 - \gamma) S^y_i S^y_{i+1}], \qquad (2.1)$$

a's and $a^\dagger$'s do not preserve this mixed set of canonical rules. However, it is possible to transform to a new set of variables that are strictly Fermi operators and in terms of which the Hamiltonian is just as simple.[1] Let

$$c_i \equiv \exp\left[\pi i \sum_1^{i-1} a_j^\dagger a_j\right] a_i$$

Main Result of "LSM" paper:
S=1/2 XY chain is solvable
by mapping to fermions

What about the LSM theorem?

APPENDIX B. NONDEGENERACY OF THE GROUND STATE AND ABSENCE OF AN ENERGY GAP IN THE HEISENBERG MODEL

We prove two exact theorems about the ground state and excitation spectrum for a Heisenberg model with nearest neighbor interactions in one dimension.

Appendix….

J. Phys.: Condens. Matter 4 (1992) 7469–7488. Printed in the UK

Hidden $Z_2 \times Z_2$ symmetry in quantum spin chains with arbitrary integer spin

Masaki Oshikawa†‡

Institute of Physics, University of Tokyo at Komaba, Komaba, Meguro-ku, Tokyo 153, Japan

Generalization of
the original AKLT state

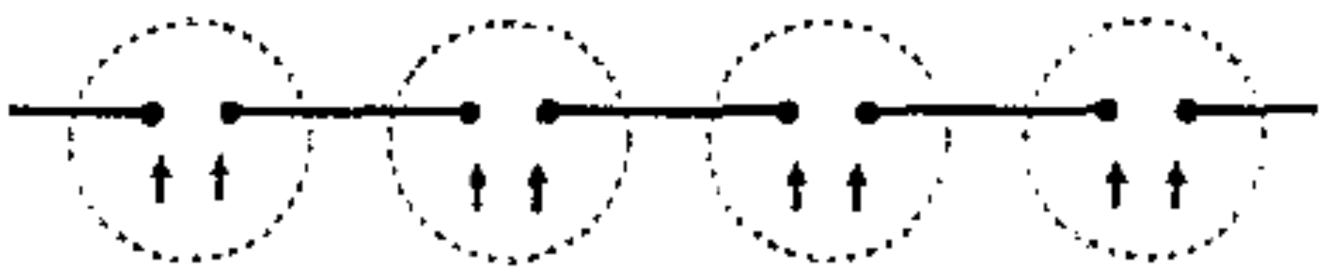
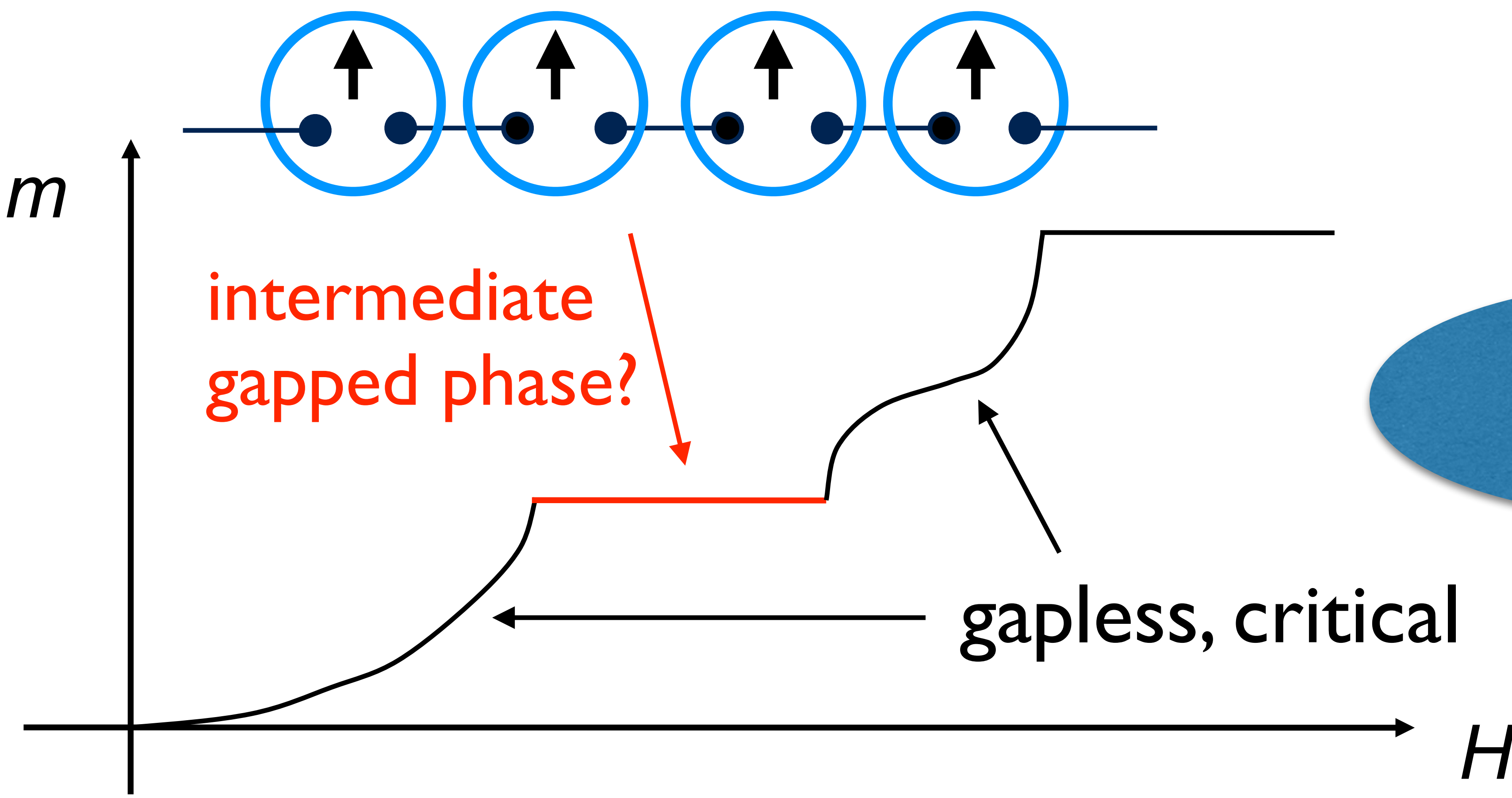


Figure 6. Ferromagnetic Ising-VBS state.

These states may represent the ground states which appear in a magnetization process of a quantum antiferromagnet. For example, let us suppose that we increase

Magnetization Process?



VOLUME 78, NUMBER 10 PHYSICAL REVIEW LETTERS 10 MARCH 1997

Magnetization Plateaus in Spin Chains: "Haldane Gap" for Half-Integer Spins

Masaki Oshikawa,[1] Masanori Yamanaka,[3] and Ian Affleck[1,2]
[1]*Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia, Canada, V6T 1Z1*
[2]*Canadian Institute for Advanced Research, University of British Columbia, Vancouver, British Columbia, Canada, V6T 1Z1*
[3]*Department of Applied Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan*
(Received 16 October 1996)

$S^z = S - 1$ $\longleftrightarrow$ particle ("magnon")

$S^z = S$ $\longleftrightarrow$ vacuum

$S^z = S - 2$

2 particles

$$\nu = S + m$$

[M.O.-Yamanaka-Affleck 1997]

"filling fraction": particle number per unit cell

$\nu = \dfrac{p}{q}$

- gapless
- ~~gapped, with unique ground state~~
- gapped, with *q*-fold degenerate ground states

valid also in higher dimensions *d*

[LSM 1961] [Affleck 1988] [M.O. 2000] [Hastings 2004]

"incommensurate filling" enforces gaplessness,
a spontaneous symmetry breaking, or a topological order (in d ≧ 2)

Recent Developments

nature physics

ARTICLES

PUBLISHED ONLINE: 14 APRIL 2013 | DOI: 10.1038/NPHYS2600

Topological order and absence of band insulators at integer filling in non-symmorphic crystals

Siddharth A. Parameswaran[1], Ari M. Turner[2], Daniel P. Arovas[3] and Ashvin Vishwanath[1,4]*

Non-symmorphic lattice with "glide symmetry":
"effective unit cell" is half of the unit cell

$$\nu_{\text{eff}} = \frac{\nu}{2}$$

filling constraints
even when $\nu \in \mathbb{Z}$

Filling constraints for spin-orbit coupled insulators in symmorphic and nonsymmorphic crystals

Haruki Watanabe[a], Hoi Chun Po[b], Ashvin Vishwanath[b,c], and Michael Zaletel[d,1]

PNAS | November 24, 2015 | vol. 112 | no. 47 | 14551–14556

Table 1. Summary of ν_{min} for elementary space groups

		Minimal filling			
ITC no.	Key elements	AI*	Ent†	Bbb‡	Manifold name
1	(Translation)	2	2	2	Torus
4	2_1	4	4	4	Dicosm
144/145	$3_1/3_2$	6	6	6	Tricosm
76/78	$4_1/4_3$	8	8	8	Tetracosm
77	4_2	4	4	4	
80	4_1	4	4	4	
169/170	$6_1/6_5$	12	12	12	Hexacosm
171/172	$6_2/6_4$	6	6	6	
173	6_3	4	4	4	
19	2_1, 2_1	8	4	8	Didicosm
24	2_1, 2_1	4	2	4	
7	Glide	4	4	4	First amphicosm
9	Glide	4	4	4	Second amphicosm
29	Glide, 2_1	8	4	8	First amphidicosm
33	Glide, 2_1	8	4	8	Second amphidicosm

*The minimal filling required to form a symmetric atomic insulator.

arXiv:1705.09298

Filling-enforced constraint on the quantized Hall conductivity on a periodic lattice

Yuan-Ming Lu,[1] Ying Ran,[2] and Masaki Oshikawa[3,4]

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[2] *Department of Physics, Boston College, Chestnut Hill, MA 02467, USA*
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[4] *Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA*

(Dated: May 29, 2017)

arXiv:1802.00587

The Lieb-Schultz-Mattis-type filling constraints in the 1651 magnetic space groups

Haruki Watanabe[1]

[1] *Department of Applied Physics, University of Tokyo, Tokyo 113-8656, Japan*

We present the first systematic study of the filling constraints to realize a 'trivial' insulator symmetric under magnetic space group $\mathcal{M}$. The filling ν must be an integer multiple of $m^{\mathcal{M}}$ to avoid spontaneous symmetry breaking or fractionalization in gapped phases. We improve the value of $m^{\mathcal{M}}$ in the literature and prove the tightness of the constraint for the majority of magnetic space groups. The result may shed light on the material search of exotic magnets with fractionalization.

Extensions of LSM-A: field of current active interest!

talks by Tasaki, Mila, ….

Summary

AKLT not only gave a strong evidence for "Haldane conjecture", but also played a seminal role in subsequent development of surprisingly many important concepts such as MPS, DMRG, SPT phases, Tensor Network states, Quantum Computation, …. in the last 30 years

Ian's numerous other contributions include
applications of CFT (Onsager Prize 2012)
and crucial extensions of the LSM theorem

Happy 65th Birthday, Ian!
We expect many more productive years!